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[Geometric Langlands ~ and ~ Derived Algebraic Geometry]

Course Notes

$\boxed{0}$ Introduction and T.O.C.

Meta-conjecture (Best hope):

$$D(\mathrm{Bun}_G(\Sigma)) \cong QC(\mathrm{Flat}_{\check{G}}(\Sigma)) \text{ as DG categories}$$

compatible w/ natural symmetries on both sides

Meta conjecture is known to be wrong as soon as $G \neq T$. Arinkin-Gaitsgory proposed a modified conj. in 2012.

Rmk) Categorical, de Rham, unramified global conjecture

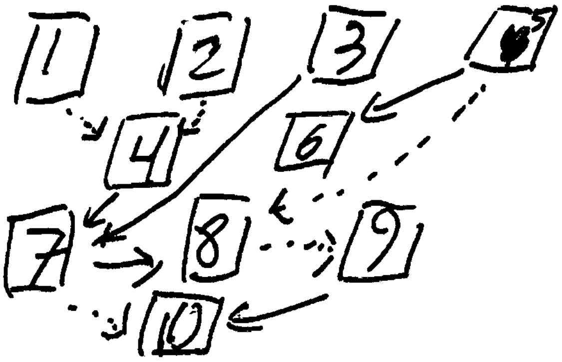
I
- $\boxed{1}$ Categorical harmonic analysis
- $\boxed{2}$ Moduli of Bundles ... Bun_G
- $\boxed{3}$ Geometric Satake ... Symmetries
- $\boxed{4}$ Localization in CFT ... example

Fundamental instances of Langlands Duality

II
- $\boxed{5}$ Derived Algebraic Geometry (DAG)
- $\boxed{6}$ Singular Support (AG)
- $\boxed{7}$ Formulation of conj

Formulation of Geometric Langlands via DAG

III
- $\boxed{8}$ Contractability
- $\boxed{9}$ Whittaker I
- $\boxed{10}$ Whittaker II



$\boxed{1}$ (1) Generalized Fourier Transform
(2) Intro to D-modules

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(1) 1. Classical Harmonic Analysis
G locally compact abelian group

Ex: $S^1, \mathbb{Z}, \mathbb{R}$

A (unitary) character of G is a homomorphism

$$\chi : G \to U(1)$$

$\hat{G} = (\{\text{characters of } G\}, \cdot)$ locally compact abelian gp.

Ex 1| $G = S^1 = [0, 2\pi]/\sim$

$e^{in(\sim)} : G \to U(1) \quad n \in \mathbb{Z}$

$\hat{G} = \mathbb{Z}$

Ex 2| $G = \mathbb{Z}$ $\quad \chi(1) \in U(1)$ determines rep.

$\Rightarrow \hat{G} = S^1$

Ex 3| $G = \mathbb{R}$ $\quad e^{itx} : \mathbb{R} \to U(1) \quad t \in \mathbb{R}$

$\hat{G} = \mathbb{R}$

Notice $\hat{\hat{G}} = G$ here.

Thm| Pontryagin duality:

$G \to \hat{\hat{G}}$ is an isomorphism

$g \mapsto \hat{g}$ where $\hat{g}(\chi) := \chi(g)$

$\hat{G}$ is Pontryagin dual

Observation: L^2 functions on G have a basis given by characters

Ex ① $f: S^1 \to \mathbb{C}$ $\quad f(\theta) = \sum_{n \in \mathbb{Z}} a(n) e^{in\theta}$ "series"

② $f: \mathbb{Z} \to \mathbb{C}$ $\quad f(n) = \frac{1}{2\pi} \int_{S^1} F(\theta) e^{in\theta}$ "discrete time"

③ $f: \mathbb{R} \to \mathbb{C}$ $\quad f(x) = \int_{\mathbb{R}} \hat{f}(t) e^{itx}$ "transform"

Thm| Plancheren $L^2(G) \cong L^2(\hat{G})$

$$e^{ixt} \leftrightarrow \delta_t$$

Rmk $G = \mathbb{R}^n$, $L^2(G) \subset S'(G)$ tempered distributions

$$S'(\mathbb{R}^n) \cong S'(\hat{\mathbb{R}}^n)$$

Obs| FT diagonalizes action of G on Fun(G)

$$\begin{tikzcd} & G \times \hat{G} \arrow[dl, "\pi_G"'] \arrow[dr, "\pi_{\hat{G}}"] & \\ G & & \hat{G} \end{tikzcd}$$

$$\begin{aligned} f(x) &= \int_{\hat{G}} F(t) e^{ixt} \\ &= \int_{\hat{G}} (\pi_{\hat{G}}^* \hat{F})(t) \chi_t(x) \, dt \\ &= (\pi_G)_* \left[\pi_{\hat{G}}^* \hat{F} \, \chi_t(x)\right] \end{aligned}$$

$G \circlearrowright G$ by translation

$y \cdot x = x + y$

$(y \cdot f)(x) = f(x - y)$

$y \cdot e^{itx} = e^{it(x-y)} = e^{-iyt} e^{ixt}$

$\{e^{ixt}\}_{t \in \hat{G}}$ eigenbasis w.r.t. translations

why?

(2) Fourier-Mukai transform:

top'l cat G, $L^2(G)$

alg. cat H, Fun(H)

$$\begin{tikzcd}
\text{Fun}(H) \arrow[d] \\
\text{sheaves}(H)
\end{tikzcd}$$

X, Y smooth alg varieties

$QC(X)$: DG category of quasi-coherent sheaves on X

Toen <u>exercise</u>| dg algebr A, A-mod dg mod
understanding derived categories

$K \in QC(X \times Y)$

$\Phi_K^{X \times Y} : QC(Y) \to QC(X)$

$\mathcal{F} \mapsto (\pi_X)_* [\pi_Y^* \mathcal{F} \otimes K]$

<u>Thm</u>| (Orlov, Toen, Ben-Zvi-Francis-Nadler)
If X, Y reasonable (colimit preserving)
then any reasonable functor $\Phi : QC(Y) \to QC(X)$
is realized by a kernel K

$G \rightsquigarrow A$ abelian variety
(connected, projective, gp variety)
$\mathbb{C}^g/\Lambda$ $g \in \mathbb{Z} > 0$

$\mu : A \times A \to A$ multiplication

$$\begin{tikzcd}
\widehat{G} & G \arrow[l, "\wedge"'] \arrow[d] & \\
 & L^2(G) \arrow[r] & QC(G)
\end{tikzcd}$$

A geometric character on A is a line bundle $\mathcal{L}$ on A s.t.

$$\mu^*\mathcal{L} \cong \mathcal{L}\boxtimes\mathcal{L} := \pi_1^*\mathcal{L}\otimes\pi_2^*\mathcal{L}$$

For $x,y\in A$

$$\mathcal{L}_{x+y}\cong\mathcal{L}_x\otimes\mathcal{L}_y$$

Rmk| previously $G \to U(1)$ homo
For $x\in G$ number

now $A \to B\mathbb{G}_m$ homo
For $x\in A$ $\mathcal{L}_x$ a line

$(\{\text{geom characters}\}, \otimes)$
is an abelian variety
"dual abelian variety" $A^\vee$

$$\begin{tikzcd} & QC(A\times A^\vee) \arrow[dl] \arrow[d] \\ A & A^\vee \end{tikzcd}$$

$\exists$ universal bundle P on $A\times A^\vee$
called poincare line bundle
s.t. $P|_{(x,\mathcal{L})} = \mathcal{L}_x$

Rmk| $\underset{\text{number}}{e^{ixt}} \leftrightarrow \underset{\text{line}}{P}$

Thm (Mukai)
$\mathbb{F}: QC(A^\vee)\to QC(A)$

$e^{ixt}\leftrightarrow \delta_t$
corresponds to

$\mathcal{L} = P|_{\mathcal{L}} \leftrightarrow \mathcal{O}_{\mathcal{L}}$ skyscraper sheaf at $\mathcal{L}$

$(\pi_1)_*(\pi_2^*\mathcal{O}_{\mathcal{L}}\otimes P)$
$= (\pi_1)_*(P_{\pi_2^{-1}\mathcal{L}})$
$= \mathcal{L}$

1) $(\delta_y * f)(\overset{\cancel{0}}{x}) = \int_{\cancel{z}\in G} \delta_y(z)\, f(x-z)$

$= f(x-y)$

2) ~~thm~~ $(L^2(G), *) \leftrightarrow (L^2(\widehat{G}), \cdot)$

$\delta_y \leftrightarrow e^{-iyt}$

$(\delta_y * (-)) \leftrightarrow (e^{-iyt}, \cdot)$

$\{e^{ixt}\}_{t\in\widehat{G}}$ spectral decomposition of ~~a~~ Fun(G)

	abelian, classical	non-abelian, cat.	plan
space of functions	$L^2(G) \simeq L^2(\widehat{G})$	$D(\mathrm{Bun}_G \Sigma) \simeq ?$	Lec 1 D Lec 2 Bun_G
operators	$G \circlearrowright L^2(G)$ translation	Sat $\circlearrowright D(\mathrm{Bun}_G)$	Lec 3 for Sat_G
eigenbasis	$\{e^{ixt}\}_{t\in\widehat{G}}$	$\{\star\}_{\star\in ?}$	Lec 4 for example

$$\begin{tikzcd} & G \times \widehat{G} \arrow[dl, "\pi_G"'] \arrow[dr, "\pi_{\widehat{G}}"] & \\ G & & \widehat{G} \end{tikzcd}$$

$\to FT$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

Fun $Y \to$ Fun (X)

$f \to (\Phi_K^{Y\to X} f)(x)$

$= (\pi_X)_*(\pi_Y^* f \cdot K)$

schwartz kernel thm.
conversely any linear op. can be realized by a kernel

X, Y smooth

$\mathrm{Hom}(C_c^\infty(F), D(Y))$

$= D(X\times Y)$

Ex/ $C^\infty(X) \xrightarrow{id} C^\infty(X)$ realised by $\delta_{diag} \in D(X\times X)$

$\mathcal{F} * \mathcal{G} = \mu_*(\mathcal{F} \boxtimes \mathcal{G})$ convolution product

$(\mathcal{F} * \mathcal{G})^\vee \simeq (\mathcal{F}^\vee \otimes \mathcal{G}^\vee)$

$(QC(A), *) \simeq (QC(A^\vee), \otimes)$

Ex: $\mathcal{O}_\alpha * \mathcal{O}_\mu = \mathcal{O}_{\alpha \otimes m}$

$\mu^* \mathcal{L} \cong \mathcal{L} \boxtimes \mathcal{L}$ $\quad \mu : A \times A \to A$

$\downarrow x$

$\Rightarrow \mu_x^* \mathcal{L} \cong \mathcal{L}_x \otimes \mathcal{L}$ $\quad \mu_x : A \to A$

$\{\mathcal{L}\}_{\mathcal{L} \in A^\vee}$ eigenbasis w.r.t. $\underbrace{\{\mu_x\}}_{\text{translations}}$

$$\begin{tikzcd}
\{\mathcal{L}\} \arrow[dd, leftrightarrow] & & \\
& QC(A^\vee) \arrow[r, shift left, "\mathbb{F}(p)"] & QC(A) \arrow[l, shift left, "\mathbb{F}(p)"] \\
\{\mathcal{O}_\mathcal{L}\} & &
\end{tikzcd}$$

$\{\mathcal{O}_\mathcal{L}\}$ skyscrapers

(3) Intro to D-modules

1. D-modules on A^1

$\mathcal{O}_X$ - module is module over $\mathcal{O}_X$

D - module is module over $\underset{\uparrow}{\mathcal{D}_X}$ differential operators

$$X = \underbrace{\mathbb{A}^1, \quad D = D(\mathbb{A}^1) = \frac{\mathbb{C}\langle x, \partial \rangle}{\partial x - x\partial = 1}}_{\text{Weyl Algebra}}$$

Rmk| This is the quantum observables for Quantum Mechanics on $\mathbb{A}^1$

"Hilbert space" $= \mathbb{C}[x]$

$$\mathcal{D} \circlearrowright \mathbb{C}[x]$$

$$x \rightsquigarrow x\cdot$$

$$\partial \rightsquigarrow \partial/\partial x$$

Goal| Find other $\mathcal{D}$-modules

f: function or distribution

$M_f = \mathcal{D}\cdot f = \mathcal{D}/\mathcal{D}P$ P is PDE for f

① Ex: $M_1 = \mathcal{D}\cdot 1 = \mathcal{D}/\mathcal{D}\partial = \mathbb{C}[x]$

② Ex. $M_{1/x} = \mathcal{D}\cdot 1/x = \mathcal{D}/\mathcal{D}(\partial x) = \mathbb{C}[x, x^{-1}]$

③ x^λ, $\lambda \in \mathbb{C}\setminus\mathbb{Z}$ $\partial(x^\lambda) = \lambda x^{\lambda-1}$

$$M_{x^\lambda} = \mathcal{D}x^\lambda = \mathcal{D}/\mathcal{D}(x\partial-\lambda) = \mathbb{C}[x,x^{-1}]x^\lambda$$

④ δ_0, $M_{\delta_0} = \mathcal{D}\delta_0 = \mathcal{D}/\mathcal{D}x = \mathbb{C}[\partial]$

⑤ $M_{e^{\lambda x}} = \mathcal{D}e^{\lambda x} = \mathcal{D}/\mathcal{D}(\partial-\lambda) = \mathbb{C}[x]e^{\lambda x}$

Note

$$0 \to M_1 \to M_{1/x} \to M_{\delta_0} \to 0$$

Rmk| In alg. geometry, D-module captures generalized functions.

e.g. $M_{e^{\lambda x}}$ D-module

consider $\mathrm{Hom}_D(M_{e^{\lambda x}}, \mathcal{O})$ = solutions to $PF=0$ in $F=\mathcal{O}$ $= \mathbb{C}e^{\lambda x}$

Claim| D is almost-commutative

$$D^{\hbar} = \frac{\langle x, \partial \rangle}{\partial x - x\partial - \hbar} = \begin{cases} D, & \hbar \neq 0 \\ \mathbb{C}[x,y] = \mathcal{O}(T^*\mathbb{A}^1), & \hbar = 0 \end{cases}$$

Filtration on D:

$$D_{\leq n} = \{ \dots \partial^{\leq n} \}$$

$$\mathrm{gr}\, D = \bigoplus_n D_{\leq n}/D_{\leq n-1} = \mathcal{O}(T^*\mathbb{A}^1)$$

D-module M might admit a filtration

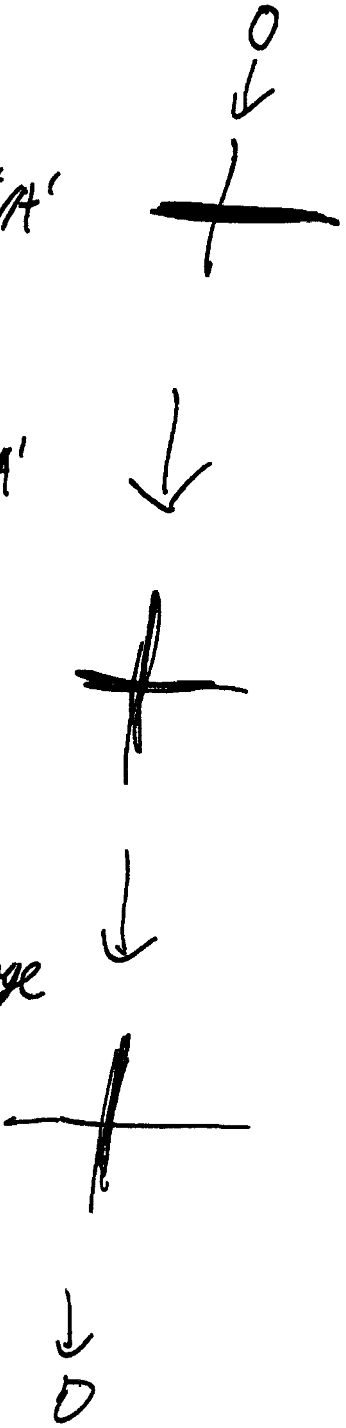
$$D_{\leq n} \cdot M_{\leq m} \subseteq M_{m+n}$$

$$\mathrm{gr}\, D \curvearrowright \mathrm{gr}\, M$$

Defn| Singular support of M (D-mod on $\mathbb{A}^1$) is support of gr M as a module over gr D

$$ss(M) \subset T^*\mathbb{A}^1$$

Ex/



① $M_1 = D/D\partial = \mathbb{C}[x]$

$\underset{\substack{\shortparallel \\ \operatorname{gr} D}}{\mathbb{C}[x,y]} \circlearrowleft \underset{\substack{\shortparallel \\ \operatorname{gr} M = m}}{\mathbb{C}[x]} \rightsquigarrow \mathbb{A}^1 \subset T^*\mathbb{A}^1$

② $M_{1/x} = D/D(\partial x) = \mathbb{C}[x,x^{-1}] \rightsquigarrow \mathbb{A}^1 \cup T_0^*\mathbb{A}^1$

D/DP $\sigma(P)$ symbol

$SS(M) = \{\sigma(P) = 0\}$

$P = \partial x$ recipe: For given P take only highest order in ∂ and change ∂ to y

$P = xy$

③ $M_{\delta_0} = D/Dx = \mathbb{C}[y]$

$P = \partial x$

④ $M_{x^\lambda} = D/(x\partial - \lambda)$

$P = x\partial - \lambda$

$\sigma(P) = xy$

$D \to$ 

Mantra for Geometric Langlands

GLC is Categorical Harmonic Analysis for D-modules on moduli space $\mathrm{Bun}_G(C)$ of G-bundles for a cpt Riemann surface, C

abelian, classical	non-abelian, categorical
$L^2(G) \cong L^2(\hat{G})$	$D(\underbrace{\mathrm{Bun}_G}_{\boxed{2}}) \simeq ?$
$G \curvearrowright L^2(G)$ translation operators	$\underbrace{\mathrm{Sat}_G}_{\boxed{3}} \curvearrowright D(\mathrm{Bun}_G)$
$\{e^{ixt}\}_{t\in\hat{G}}$ eigenbasis	$\{★\}_{★\in ?} \leftarrow \boxed{4}$ eigenbasis

$\boxed{2}$ Moduli of Bundles (1)
and Hitchin Fibration (2)

(1) 1. Line bundles

X variety

Picard variety of line bundles $\underline{\mathrm{Pic}(X)}$

$$= H^1(X, \underbrace{\mathcal{O}_X^\times}_{\substack{\text{nowhere-vanishing}\\ \text{functions on } X}})$$

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 0$$

$$\begin{tikzcd}
\cdots \arrow[r] & H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathcal{O}_X) \arrow[r] & \overset{S}{\overset{\cap}{H^1(X,\mathcal{O}_X^*)}} \arrow[dll, "c = \deg"'] \\
& H^2(X,\mathbb{Z}) \arrow[r] & \cdots &
\end{tikzcd}$$

$\underline{\mathrm{Pic}}^0(X) = \ker(\deg : H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z})$

$= H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$

<u>Ex</u> $X = C$ cpt. Riemann surface of genus g

$\underline{\mathrm{Pic}}^0(X) = \mathbb{C}^g/\mathbb{Z}^{2g}$ g-dimensional abelian variety

$\underline{\mathrm{Pic}}^0(X)$ is an abelian variety

$$A \overset{\underline{\mathrm{Pic}}^0(X)}{\overset{\shortparallel}{=}} H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$$

$A = V/\Lambda \leftrightarrow A^\vee = V^*/\Lambda^*$

$A^\vee = H^0(X,\Omega_X)^*/H^1(X,\mathbb{Z}) = \mathrm{Alb}(X)$ "Albanese variety"

Ω_X is sheaf of 1-forms

Fix $x_0 \in X$

$u : X \to \mathrm{Alb}(X)$ Albanese map

$$x \to \Big(\int_{x_0}^{x} : \underset{\underset{H^0(X,\Omega_X)}{\cap}}{\omega} \mapsto \int_{x_0}^{x} \omega$$

Ex 1/ $X = C$

AJ_{x_0} : $C \hookrightarrow Alb(C) = Jac(C)$ "Abel Jacobi"

$$\begin{tikzcd}
H^1(X, \mathcal{O}_X) \arrow[r, phantom, "\cong"] & H^0(X, \Omega_X)^* \\
H^1(C, \mathbb{Z}) \arrow[u, leftrightarrow] \arrow[r, phantom, "\cong"] & H_1(C, \mathbb{Z}) \arrow[u, hook]
\end{tikzcd}$$

$\Rightarrow \underline{Pic}^0(C) \cong Jac(C)$

~~Jac C~~ ~~$A^\vee = Jac\ C$~~

Rmk/

$$\begin{tikzcd}
\pi_1(AT_{x_0}) : \pi_1(C) \arrow[r] \arrow[dr] & \pi_1(Jac\ C) \\
& \pi_1^{ab}(C) = H_1(C, \mathbb{Z}) \arrow[u, hook]
\end{tikzcd}$$

Note local system $\cong$ rep of π_1 $\cong$ flat connection

$$\begin{tikzcd}
\text{Given: } \pi_1(C) \arrow[r] \arrow[dr] & \mathbb{C}^* \\
& \pi_1(C)^{ab} \arrow[u] \arrow[d, equal] \\
& \pi_1(Jac(C))
\end{tikzcd}$$

rk 1
flat connection on C

we get rk 1 flat connection on $Jac(C)$

$$\begin{tikzcd}
D(Bun_G(C)) \arrow[r, phantom, "\cong"] & QC(Flat_{\check{G}}\ C) \\
Jac \arrow[u] & Flat_{GL_1} \arrow[u] & \text{skyscraper sheaf} \arrow[ul, bend right]
\end{tikzcd}$$

$G = GL_1$

D-module on
Jac C

$A = Jac\ C$ $\quad A^\vee = Jac\ C$

$$QC(A) \underset{FM}{\cong} QC(A^\vee)$$

Line bundle on A $\{ \mathcal{L} \leftrightarrow \mathcal{O}_{\mathcal{L}} \}$ skyscraper on $A^\vee = \underline{Pic^0}(A)$

$$Jac = Jac\, C = H^0(C, \Omega_C)^* / H_1(C, \mathbb{Z})$$

$$A = T^* Jac = Jac \times H^0(C, \Omega_C)$$

$$\begin{tikzcd}
A = T^* Jac = Jac \times H^0(C, \Omega_C) \arrow[d, "\text{fiber is Jac}"'] & A^\vee = Jac \times H^0(C, \Omega_C) = T^* Jac \arrow[dl] \\
B = H^0(C, \Omega_C) &
\end{tikzcd}$$

$$QC(A) \simeq QC(A^\vee)$$

$$\dim T^* Jac = \dim Jac + \dim B$$

$$2g = g + g$$

$T^* Jac$ is a symplectic space

Jac is Lagrangian

$$\mathbb{C}[B] = \mathbb{C}[T^* Jac]$$

$\Rightarrow T^* Jac \to B$ is integrable system

Def (M^{2n}, ω)

$H \in Fun(M)$ is completely integrable

if $\exists F_1 = H, F_2, \dots, F_n$ s.t.

- $dF_1 \wedge \dots \wedge dF_n \neq 0$
- $\{F_i, F_j\} = 0 \quad \forall i, j$

Hitchin's integrable system

Rmk (T-duality)

When one has a family of abelian varieties over B, one can construct

$A^\vee$ Fiberwise

$$A \overset{\text{T-dual}}{\longleftrightarrow} A^\vee$$

2. Moduli of bundles, flat bundles, Higgs bundles

$Bun_G = Bun_G(C)$

$$\begin{tikzcd} G \circlearrowright & P \arrow[d] \\ & C \end{tikzcd}$$

moduli space of G-bundles on C

$D(Bun_G)$ Note $D(X) \sim QC(T^*X)$

$\sim QC(T^*Bun_G)$

$T_P \, Bun_G(C) = H^{0,1}(C, ad P)$

$G = GL_n$ $\quad A \in T_E \, Bun_n(C) = H^{0,1}(C, End\, E)$

$$\bar\partial_E \rightsquigarrow \bar\partial_E + [A, \cdot]$$

$T^*_P \, Bun_G(C) = H^0(C, \Omega_C \otimes ad\, P)$

$T^* Bun_G(C) = \{(P, \varphi) \mid \varphi \in H^0(C, \Omega_C \otimes ad P)\}$

$\|$ $\qquad\qquad \uparrow$

$Higgs_G(C)$ $\qquad$ Higgs field

$G = GL_1$

$E = \mathcal{L}$

$\mathrm{End}\, E = \mathrm{End}\, \mathcal{L} = \mathcal{L}^* \otimes \mathcal{L} = \mathcal{O}_C$

$\varphi \in H^0(C, \Omega_C \otimes_{\mathcal{O}_C} \mathcal{O}_C)$

$= H^0(C, \Omega_C) = \mathcal{B}$

Flat C

$\nabla : E \to \Omega_C \otimes E$

$\nabla(fs) = df \otimes s + f \nabla s$

where $f \in \mathcal{O}_C$, $s \in E$

$(\nabla_1 - \nabla_2)(fs) = f(\nabla_1 - \nabla_2)(s)$

$\nabla_1 - \nabla_2 \in H^0(C, \Omega_C \otimes \mathrm{End}\, E)$

Flat_G is an affine bundle modelled on

$T^* \mathrm{Bun}_G \to \mathrm{Bun}_G$

Betti moduli

<u>Rmk</u> $\mathrm{Flat}_G(C) \sim \mathrm{Loc}_G(C) = \mathrm{Hom}(\pi_1(C), G)/G$

analytically equivalent

character variety

de Rham moduli

but NOT algebraically

$$\begin{tikzcd} T^* \mathrm{Bun}_G \arrow[dr, bend left] & & \mathrm{Flat}_G \arrow[dl, dashed, "?\ \text{not always}"] \\ & \mathrm{Bun}_G \arrow[ul, bend left] & \end{tikzcd}$$

Rmk 1 $\mathcal{M}_H \simeq (T^* \mathrm{Bun}_G)^{st}$

Hitchin moduli (soln to Hitchin eqn)

Hyper käler manifold I, J, K cpx structure

$(\mathcal{M}_H, I) \sim T^* \mathrm{Bun}_G$

$(\mathcal{M}_H, \text{any other}) \sim \mathrm{Flat}_G$

(2) Hitchin System

1. Spectral Correspondence

Idea: Understand a linear map by its spectrum

V: cpx vector space $\dim_{\mathbb{C}} V = n$

IF $\varphi: V \to V$ is generic

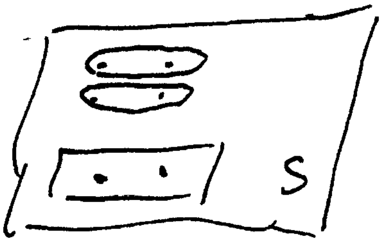
then φ has eigs $\lambda_1 \dots \lambda_n$ $\lambda_i \neq \lambda_j$ for $i \neq j$

$\lambda_i \rightsquigarrow L_i$ eigenspace

$V = L_1 \oplus \cdots \oplus L_n$

Generalizations

① Introduce a parameter space



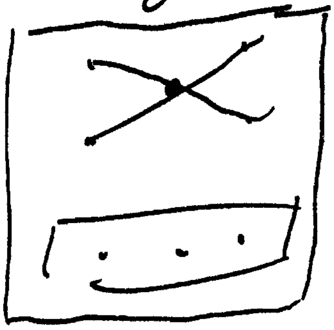
$\phi: S \to \mathrm{End}\, V$

$s \to \varphi_s$ is generic

$$\begin{tikzcd} S \times \mathbb{C} \supset \bar{S} = \{(s,\lambda) \mid \lambda \text{ eig of } \varphi_s\} \arrow[d, "n:1 \text{ cover}"] \\ S \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{L}_{(s,\lambda)} \arrow[r] \arrow[d] & \mathcal{L} \subset \bar{S} \times V \arrow[d] \\ (s,\lambda) \arrow[r] & s \end{tikzcd}$$

② allow Φ_s to have repeated eigs



Note

Note: φ_s is as generic as possible

$$\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} << \boxed{\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}} << \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

For now, use Jordan blocks with all 1s in the superdiagonal

$$\begin{tikzcd} \bar{S} \arrow[d, "\text{branched cover}"] & \mathcal{L} \arrow[d, "\text{line bundle}"] \\ S & \bar{S} \end{tikzcd}$$

$$\chi : \operatorname{End} V \to \mathbb{C}^n$$
$$\varphi \to (a_1(\varphi), \dots, a_n(\varphi))$$

where $\det(t \cdot id_V - \varphi) = t^n - a_1(\varphi)t^{n-1} + \dots + a_n(\varphi)$

$$\begin{tikzcd} \bar{\mathbb{C}}^n = \{(a_1, \dots, a_n, t)\} \in \mathbb{C}^n \times \mathbb{C} \arrow[d] \\ \mathbb{C} \end{tikzcd}$$

Exer

$$\begin{tikzcd} \bar{S} \arrow[r] \arrow[d] & \bar{\mathbb{C}}^n \arrow[d] \\ S \arrow[r, "\chi\circ\varphi"] & \mathbb{C}^n \end{tikzcd}$$

③ $GL(V) \circlearrowright \operatorname{End}(V)$

w/ $G \circlearrowright \mathfrak{g}$

④ Replace V by a vector bundle E on S

$\rightsquigarrow \phi \in H^0(S, \mathrm{End}\, E)$

⑤ Introduce coefficient object K

$$V \to K \otimes V$$

$\rightsquigarrow \varphi \in H^0(S, K \otimes \mathrm{End}\, E)$

K is rk r vector bundle on S

$S = X$ cpx alg. variety

On $\mathcal{U} \subset X$, $K|_{\mathcal{U}} \simeq \mathbb{C}^r \otimes \mathcal{O}_{\mathcal{U}}$

$\varphi|_{\mathcal{U}} = (\varphi_1, \ldots, \varphi_r)$ $\varphi_i \in H^0(\mathcal{U}, \mathrm{End}\, E)$

spectral cover construction fails

<u>Defn</u>| A K-valued Higgs field is $\varphi \in H^0(X, K \otimes \mathrm{End}\, E)$ s.t. $\phi \wedge \phi = 0 \in H^0(X, \wedge^2 K \otimes \mathrm{End}\, E)$

$\phi : E \to K \otimes E$

$\Leftrightarrow \overset{*}{\phi^t} : K^\vee \otimes E \to E$

$\Leftrightarrow \mathrm{Sym}^\bullet K^\vee \otimes E \to E$

$$\begin{tikzcd} \mathrm{Sym}^\bullet K^\vee \arrow[d, equal, "!!"] \\ \mathcal{O}_Y \\ Y = \mathrm{tot}(K) \arrow[d, "\pi"] \\ X \end{tikzcd}$$

$\mathcal{O}(V) = \mathrm{Sym}\, V^*$

$$\begin{tikzcd} \mathcal{E} & E \\ \pi_* : \{\text{quasi-coh sheaves on } X\} \arrow[r, "\simeq"] & \{\text{quasi-coh Higgs sheaves on } X\} \\ \{\text{coh sheaves on } Y \text{ wt finite support on } X\} \arrow[u] \arrow[r, "\simeq"] & \{\text{coh. Higgs sheaves on } X\} \arrow[u] \end{tikzcd}$$

Hitchin System

$S = X = C$

$K = \Omega_C$

$Y = \mathrm{tot}(K) = T^*C$

$$\left\{ \begin{array}{c} \text{coh sheaves on } T^*C \\ \text{w/ support finite on } C \end{array} \right\} \overset{\pi_*}{\simeq} \left\{ \begin{array}{c} \text{coh. Higgs} \\ \text{sheaves on } C \end{array} \right\}$$

$Y \to X$

$(E, \varphi) \in T^*\mathrm{Bun}_n \leftarrow GL(n)$ $\quad \varphi_x : E_x \to T^*_x C \otimes E_x$

$\varphi : E \to \Omega_C \otimes E$

$\mathcal{O}_{T^*C} \circlearrowright \mathcal{E}$

$\mathrm{supp}\, \mathcal{E} = \Sigma \subset T^*C$

$= \{(x,\lambda) \mid \lambda \text{ an eig of } \varphi_x\}$

$\dim V = n$	$\overline{S} \subset S \times \mathbb{C}$ $\downarrow n:1$ S	$\Sigma \subset T^*C$ $\downarrow n:1$ C	$\mathrm{rk}\, E = n$
	$\mathcal{L}$ $\downarrow$ $\overline{S}$	$\mathcal{E}$ $\downarrow$ Σ	

$\chi : \mathrm{End}\, V \to \mathbb{C}^n$

$\chi_H : T^*\mathrm{Bun}_n \to \mathcal{B} \simeq \{\text{spectral curves}\}$

$\downarrow$

$(E, \varphi) \mapsto \mathrm{supp}\, \mathcal{E} = \Sigma$

is

$\mathcal{E}$

Given φ

$$\varphi \to \lambda^n - tr(\varphi)\lambda^{n-1} + \cdots + (-1)^n \det(\varphi)$$

$$tr\,\varphi \in H^0(C, \Omega_C)$$

$$\det\varphi \in H^0(C, \Omega_C^{\otimes n})$$

$$\Rightarrow u \in \mathcal{B} = \bigoplus_{k=0}^{n} H^0(C, \Omega_C^{\otimes k})$$

$$\Sigma_u = \{(\lambda, t) / \lambda^n - u_1\lambda^{n-1} + \cdots + (-1)^n u_n = 0\}$$

	abelian	non-abelian
dualit	$\begin{tikzcd} T^*\mathrm{Jac} \arrow[dr] & & T^*\mathrm{Jac} \arrow[dl] \\ & B = H^0(C, \Omega_C) & \end{tikzcd}$	$\begin{tikzcd} T^*\mathrm{Bun}_n \arrow[dddrr, "\chi_H"'] & \mathcal{A} \arrow[l, hook'] & & \mathcal{A}^\vee \arrow[r, hook] \arrow[ddl] & T^*\mathrm{Bun}_n \arrow[dddll, "\chi_H"] \\ & \mathrm{Pic}\,\Sigma_u \arrow[u] \arrow[d] & & & \\ & u \arrow[r] & \mathcal{B}^{reg} \arrow[d] & & \\ & & \mathcal{B} & \text{``Hitchin Base''} \arrow[l] & \end{tikzcd}$
n	$QC(T^*\mathrm{Jac}) \circlearrowleft$	$QC(\mathcal{A}) \simeq QC(\mathcal{A}^\vee)$
	$\mathbb{C}[B] = \mathbb{C}[T^*\mathrm{Jac}]$	$\mathbb{C}[\mathcal{B}] = \mathbb{C}[T^*\mathrm{Bun}_n]$ Hitchin s system

$$\mathcal{B} \supset \mathcal{B}^{reg} = \{u \in \mathcal{B} / \Sigma_u \text{ smooth}\}$$

$$\begin{tikzcd} T^*\mathrm{Bun}_n \arrow[r] & \mathcal{B} \\ \mathrm{Pic}\,\Sigma_u \arrow[u, phantom, sloped, "\subset"] \arrow[r] & u \in \mathcal{B}^{reg} \end{tikzcd}$$

Rmk/ This picture can be generalized to $G \Leftrightarrow \check{G}$

Donagi–Pantev

Rmk/ $G = SL_2 \longrightarrow H^0(C, \Omega_c^{\otimes 2}) \Leftrightarrow \check{G} = PGL_2$

$GL_2 \to \bigoplus_{k=1}^{2} H^0(C, \Omega_c^{\otimes k})$

space of quadratic differentials

√ higher teichmuller theory

4d $N=2$ SUSY gauge thry

Rmk]

$$\begin{tikzcd} \Sigma \arrow[d] & \subset T^*C \\ C & \end{tikzcd}$$

Seiberg witten (IR curve)

Gaiotto curve (UV curve)

$\mathcal{B}$ coulomb branch of 4d-thry
from compactification of thory X along C

$\mathcal{M}_H$ coulomb brangch of of 3d-thry
from further compactification along S^1

$\boxed{3}$ Geometric Satake Equivalence I

Motivation and Preliminary

(1) What is $\check{G}$? "Langlands dual group"

W. Thurston "On progress and proof of mathematics"

"One person's clear mental image is another's intimidation"

Goal of class is to parse this sentence:

"$\check{G}$ is the group of automorphisms of cohomology of $G(\mathbb{C}[t])$-equivariant perverse sheaves on $\underbrace{G(\mathbb{C}(t))/G(\mathbb{C}[t])}_{\text{affine Grassmannian}}$"

↗ Tannakian formalism

↘ Idea of Hecke algebras

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1. Tannakian formalism

Previously, $G^{\text{loc. cpt}}_{\text{abelian}} \rightsquigarrow \widehat{G} \rightsquigarrow \widehat{\widehat{G}} = G$

Now, G affine alg. gp. $\rightsquigarrow$?

For $G^{\text{cpt.}}_{\text{non-abelian}} \overset{\leftarrow \text{Tannaka-Krein duality}}{\rightsquigarrow}$ cat. of unitary characters

consider $\operatorname{Rep} G$ cat of f.d rep'ns of G over k

- $\operatorname{Rep} G$ is k-linear abelian category (hom-space is k-linear vector space)
- $\operatorname{Rep} G$ is monoidal: $\otimes : \operatorname{Rep} G \times \operatorname{Rep} G \to \operatorname{Rep} G$
  $V, W \to V \otimes W$

w/ associativity, unity. $k = \mathbb{1} \in \operatorname{Rep} G$

- Rep G is symmetric monoidal,

  $\exists\, \gamma_{V,W} : V \otimes W \cong W \otimes V$ s.t. $\gamma_{WV} \circ \gamma_{VW} = id_{V\otimes W}$
  (canonical)

- Rep G is <u>rigid</u>

  $\forall V \in \text{Rep } G, \exists V^* \in \text{Rep } G$

$\exists$ forgetful functor

$\omega : \text{Rep } G \to (\text{Vect}_k, \otimes)$ ↙ symmetric monoidal rigid

exact, fully faithful ↖ fiber functor

<u>Defn</u> A <u>neutral Tannakian category</u> is a rigid symmetric monoidal k-linear abelian category A equipped with a fiber functor

$G \rightsquigarrow \text{Rep } G$ neutral Tannakian cat.

<u>Thm</u> (Tannakian Formalism)

A neutral tannakian category $(A, \omega : A \to \text{Vect}_k)$ is equivalent to Rep G

Where $G = \text{Aut}^{\otimes}(\omega)$

<u>Ex</u>

① $\text{Vect}_k \xrightarrow{\omega = id} \text{Vect}_k \rightsquigarrow \text{Aut}(id) = \{1\}$

$V \to V \qquad \text{Vect}_k \cong \text{Rep}\{1\}$

② $\text{Rep } G \xrightarrow{\omega = \text{Forget}} \text{Vect}_k$

$V \longmapsto V \qquad \text{Aut}(\omega) = G$

$\text{Rep } G \cong \text{Rep } G$

③ $\overset{\text{f.d.}}{\text{Vect}_k^{\mathbb{Z}}} \longmapsto \overset{\text{f.d.}}{\text{Vect}_k}$ $\quad \text{Aut}(\omega) = k^{\times}$

$\{V_n\}_{n \in \mathbb{Z}} \longmapsto V = \oplus V_n \qquad \rightsquigarrow \text{Vect}_k^{\mathbb{Z}} \cong \text{Rep}(k^{\times})$

~~gr~~ 1-gr 2-gr

$V_1 \otimes V_1 \cong V_2$

$z \in k^*$ acts as z^n on V_n

④ $LS(X)$ cat of local systems on X, $x \in X$

$$\begin{tikzcd}
\pi_1(X,x) \arrow[r, phantom, "\circlearrowright"] & LS(X) \arrow[d, "\text{Fiber at } x"] \\
& \mathrm{Vect}
\end{tikzcd}$$

$\rightsquigarrow$ $LS \overset{?}{\simeq} \mathrm{Rep}\ \pi_1(X,x)$

$LS \simeq \mathrm{Rep}\ \pi_1^{alg}(X,x)$

alg hull of π_1

Rmk main motivation for <u>motives</u>

$\rightsquigarrow$ descendants $\left(\begin{array}{l} \text{Galois rep'ns } (\ell\text{-adic}) \\ \text{mixed Hodge structure} \end{array} \right)$

Rmk (extensions)

- $A \to QC(S)$
- A symm monoidal $\overset{E_\infty}{\rightsquigarrow}$ $A = \mathrm{Rep}\, G$
- A braided (E_2) monoidal $\rightsquigarrow$ $A \cong \mathrm{Rep}\, U_q G$
- A E_n-monoidal $\rightsquigarrow$ $A \simeq ?$
- In DAG,

  $A \to \mathrm{Vect}_k$

  replaced by $X \overset{\leftarrow\cdots}{\longmapsto} QCoh(X)$

  If $X = BG = pt/G$

  $QCoh(BG) \cong \mathrm{Rep}\, G$

2. Hecke algebra

$G \rightsquigarrow (\mathcal{A}_{G,\omega})$ neutral Tannakian

$\rightsquigarrow \mathcal{A}_G \simeq \mathrm{Rep}(\mathrm{Aut}^{\otimes}(\omega))$

where $\mathrm{Aut}(\omega) = \check{G}$

<u>Goal</u>: Construct $\mathcal{A}_G$

Let's work with a finite gp. H

$(\mathbb{C}[H], *)$ group algebra

$$(\varphi_1 * \varphi_2)(h) = \sum_{x \in H} \varphi_1(x)\, \varphi_2(x^{-1}h) = \sum_{xy=h} \varphi_1(x) \varphi_2(y)$$

$$\begin{tikzcd} & H \times H \arrow[dl, "\pi_1"'] \arrow[d, "m"] \arrow[dr, "\pi_2"] & \\ H & H & H \end{tikzcd}$$

$$(\varphi_1 * \varphi_2) = m_*(\pi_1^* \varphi_1 \cdot \pi_2^* \varphi_2)$$

Where for $f : X \to Y$

$$(f^* \psi)(x) = \psi(f(x)), \quad (f_* \varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x)$$

Use $K \subset H$ subgp

to find a comm alg.

$${}^K\mathbb{C}[H]^K = \mathbb{C}[K \backslash H / K] =: \overset{\substack{\text{Hecke algebra} \\ \downarrow}}{\mathcal{H}_{H,K}}$$

$$\begin{tikzcd} & K\backslash H \times_K H/K \arrow[dl, "\pi_1"'] \arrow[d, "m"] \arrow[dr, "\pi_2"] & \\ K\backslash H/K & K\backslash H/K & K\backslash H/K \end{tikzcd}$$

$$\varphi_1 * \varphi_2 = m_*(\pi_1^* \varphi_1 \cdot \pi_2^* \varphi_2)$$ associative alg.

Prop For any rep'n V of H

$$^K V : \{ v \in V \mid k.v = v \} \circlearrowleft \underset{\substack{\uparrow \\ \text{universal way}}}{\mathcal{H}_{H,K}}$$

"Recall"

Frobenius reciprocity

$$\mathrm{Hom}_H(\mathrm{Ind}_K^H W, V) = \mathrm{Hom}_K(W, \mathrm{Res}_H^K V)$$

$$\mathrm{Res}_H^K : \mathrm{Rep}\, H \to \mathrm{Rep}\, K$$

$$\mathrm{Ind}_K^H : \mathrm{Rep}\, K \to \mathrm{Rep}\, H$$

$$W \to \mathbb{C}[H] \otimes_{\mathbb{C}[K]} W$$

"pf"
$$\begin{aligned} {}^K V &= \mathrm{Hom}_K(\mathbb{C}_{\mathrm{triv}}, \mathrm{Res}_H^K V) \\ &= \mathrm{Hom}_H(\mathrm{Ind}_K^H \mathbb{C}_{\mathrm{triv}}, V) \\ &= \mathrm{Hom}_H(\mathbb{C}[H/K], V) \circlearrowleft \mathcal{H}_{H,K} \end{aligned}$$

pre composition

$$\mathrm{End}_H(\mathbb{C}[H/K]) \overset{!}{=} \mathcal{H}_{H,K}$$

$$\begin{array}{ccc} \| & & \| \\ {}^K\mathbb{C}[H/K] & = & \mathbb{C}[K\backslash H/K] \end{array}$$

Rmk If K is small, Rep H can survive ${}^K(-)$, so $\mathcal{H}_{H,K}$ knows a lot about Rep H

If K is large, $K \backslash H/K$ is small so $\mathcal{H}_{H,K}$ has better structure (e.g. commutativity)

For Rep G one needs to find the right balance!

Let's translate all this into geometry

$$\begin{tikzcd} & X \times H \arrow[dl, "\pi_1"'] \arrow[d, "a"] \arrow[dr, "\pi_2"] & \\ X & X & H \end{tikzcd} \rightsquigarrow \begin{tikzcd} & X \times_K H/K \arrow[dl, "\pi_1"'] \arrow[d, "a"] \arrow[dr, "\pi_2"] & \\ X/K & X/K & K\backslash H/K \end{tikzcd}$$

$\varphi \in \mathbb{C}[X]$, $\alpha \in \mathbb{C}[H]$

$\varphi \cdot \alpha = a_*(\pi_1^* \varphi \cdot \pi_2^* \alpha)$

$\mathbb{C}[X/K] \circlearrowleft \mathcal{H}_{H,K}$

$$\underset{\overset{\shortparallel}{D(Bun_G)}}{\underline{Fun(X/K)}} \circlearrowleft \mathcal{H}_{H,K} = Fun(K\backslash H/K)$$

$Sat_G \circlearrowright D(Bun_G)$

Goal: Find X, H, K

s.t. $\cdot\ X \circlearrowleft H$

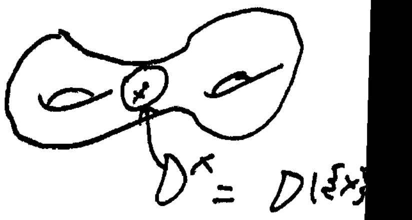
$\cdot\ X/K = Bun_G = Bun_G C$

$\rightsquigarrow D(Bun_G) \circlearrowleft D(K\backslash H/K)$

First work in top'l cat. G connected Lie group

C cmpct. Riemann of genus g.

prop $Bun_G^{top}(C) \cong L \backslash L^{top}G / L_+^{top} G$



where $L^{top}G = \{D^\times \to G\}$

$L_+^{top} G = \{D \to G\}$

$L_{out} G = \{C^\times \to G\}$

$D =$ disk around x

$D^\times = D \backslash \{x\}$

$C^\times = C \backslash \{x\}$

pf) $L^{top}G \cong (P, \alpha, \beta) \Leftarrow$ claim

where P is a G-bundle on C

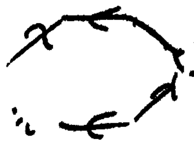
$\alpha : P|_D \cong P^0|_D$

$\beta : P|_{C^\times} \cong P^0|_{C^\times}$

trivialization

D is contractible ⇝ α exists

$P|_{C^\times}$



$\cong \vee_{2g} S^1 \quad [S^1 \to BG]$

$= \pi_1(BG) = \pi_0(G)$

homotopy class is trivial for each S^1
so the whole bundle is trivial

$\gamma = \beta \times \alpha^{-1}_{D^\times} : P^0|_{D^\times} \to P^0|_{D^\times} = D^\times \times G$

$D^\times \to G \in L^{top}G$

$\Rightarrow \eta \in L^{top}G$, triv on D
triv on $C^\times$

glue them using η to get P

$L^{top}_+ G$ = trivializations on D

$L^{top}_{out} G$ = trivializations on $C^\times$

Goal: $Bun_G = X/K$

$X = L^{top}_{out}G \backslash L^{top}G$, $H = L^{top}G$

$K = L^{top}_+ G$

In an algebraic category:

Thm1 (Weil Uniformization)

C smooth proper curve $/ k = \mathbb{F}_q$

$$Bun_G^{(C)}(k) \cong G(k(C))\backslash G(\mathbb{A})/G(\mathbb{O})$$

where $Bun_G = Bun_G(C)$ is moduli of G-bundles on C, $k(C)$ is function field

adele ↘ $\mathbb{A} = \prod^{res}_{x\in C} K_x$, $\quad \mathbb{O} = \prod_{x\in C} \mathcal{O}_x$

with $K_x = k((t_x))$ $\quad \mathcal{O}_x = k[[t_x]]$

Rmk1 $L^2(G(k(C))\backslash G(\mathbb{A})/G(\mathbb{O}))$

is space of automorphic representations!
(for unramified case)

$$\begin{array}{ccc} G(k(C))\backslash\, G(K_x) & \circlearrowleft & G(K_x) \\ & & \cup \\ & & G(\mathcal{O}_x) \end{array}$$

$$D(G(\mathcal{O}_x)\backslash G(K_x)/G(\mathcal{O}_x)) \circlearrowleft D(Bun_G) \qquad \forall x \in C$$

$\quad \Vert$

$Sph_{G,x}$ spherical Hecke category

We finally understand the sense of spectral Decomposition of $D(Bun_G)$

$\check{G}$?

try to find a neutral Tannakian category $\mathcal{A}_G$

Defn $Gr_G = G(\mathcal{K})/G(\mathcal{O})$ affine Grassmannian

$\mathcal{K} = \mathbb{C}((t)),\ \mathcal{O} = \mathbb{C}[[t]]$

$$\boxed{\begin{array}{l} \mathbb{D} = \operatorname{Spec} \mathcal{O} \\ \mathbb{D}^\times = \operatorname{Spec} \mathcal{K} \end{array}}$$

$G(\mathcal{K})$ "=" $\{\text{Maps } \mathbb{D}^\times \to G\}$ loop group

$G(\mathcal{O})$ "=" $\{\text{Maps } \mathbb{D} \to G\}$ affine Kac-Moody group

like the flag variety G/B

Lustig, Drinfeld, Ginzburg, Mirkovic-Vilonen

Thm (Geometric Satake)

$\mathcal{P}_{G(\mathcal{O})}(Gr_G)$ is a neutral Tannakian category

$$\mathcal{P}_{G(\mathcal{O})}(Gr_G) \cong \operatorname{Rep} \check{G}$$

Rmk $\mathcal{P}$ abelian category of perverse sheaves

D-modules $\overset{RH}{\sim} \underset{\substack{\uparrow \\ \text{works better} \\ \text{in char} > 0}}{\mathcal{P}}$

$$\begin{array}{c} D(G(\mathcal{O}_x)\backslash G(\mathcal{K}_x)/G(\mathcal{O}_x)) \\ \cup \\ \mathcal{P} \end{array}$$ has natural $*$ structure

Miracles]

① $\mathcal{P}$ is closed under $*$!

② $\phi_1, \phi_2 \in \mathcal{P}$

$$\phi_1 * \phi_2 \cong \phi_2 * \phi_1$$

2. Basic Geometry of Gr_G

$G = GL_n$

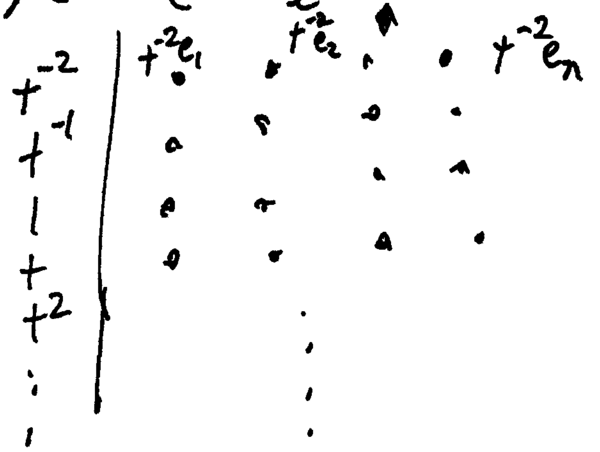
a lattice K^n is an $\mathcal{O}^n$ submodule L s.t. $t^N \mathcal{O}^n \subset L \subset t^{-N} \mathcal{O}^n$ for some N

prop] $Gr_G \cong \{\text{Lattices in } K^n\}$

$G(K) \curvearrowright \mathcal{O}^n$ lattice transitive

$G(\mathcal{O})$: stabilizer $\rightsquigarrow Gr_G = \{\text{lattice}\}$

K^n $\{t^i e_l \mid i \in \mathbb{Z}, l = 1, \cdots, n\}$



Imagine $K = \mathbb{Q}$, $\mathcal{O} = \mathbb{Z}$ $\frac{1}{2^2}\mathbb{Z} \subset \mathbb{Q}$

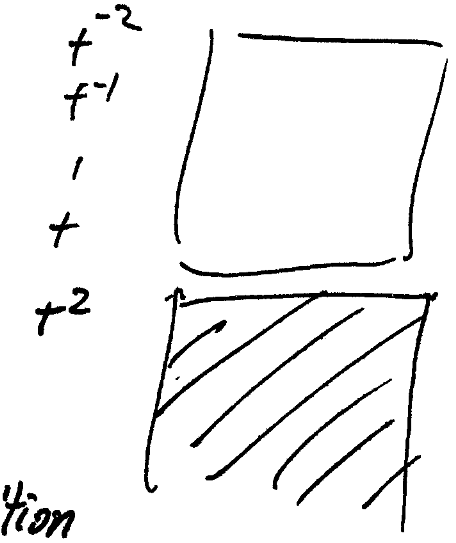
$\mathcal{O}^n$ submodule $\leftrightarrow$ closed under t.

$Gr_G^k \subset \mathbb{C}^{2kn}$

$\shortparallel$

$\{$can take $N=k\}$

$k=2$



closed condition

$Gr_G = \bigcup^{k}_{k\to\infty} Gr_G^k$ ind-projective variety

Quantization of Hitchin Systems

(1) Intro to Geometric Representation Theory

2) A slice of Geometric Langlands Correspondence

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1. Borel-Weil Theorem

let G connected, semisimple Lie group$/\mathbb{C}$

Ex $SL_n = SL_n(\mathbb{C})$

Want to understand representations of G

⇝ Consider a flag manifold, $\mathcal{B}$ the set of Borel subgroups of G

Ex $\mathcal{B} = \{\text{Upper triangular matrices}\}$

$G \curvearrowright \mathcal{B}$ by $g \cdot B = g \cdot B \cdot g^{-1}$

Prop (1) $G \curvearrowright \mathcal{B}$ transitive

(2) $N_G(B) = B$

$(\{g \in G : g \cdot B = B\} = B)$

$\Rightarrow \quad \mathcal{B} = G/B$ algebraic variety

Ex $G = SL_2 \curvearrowright \mathbb{C}^2$

$B \leftrightarrow$ stabilizer of a line

$\mathcal{B} \cong \{\text{lines in } \mathbb{C}^2\} \cong \mathbb{P}^1$

G/B with $G = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| ad-bc=1\right\}$

$B = \left\{\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}\right\}$

$\|$

$$\Rightarrow \left\{\begin{pmatrix} z_1 & * \\ z_2 & * \end{pmatrix} \middle| \begin{matrix} (z_1, z_2) \neq 0 \\ (z_1, z_2) \sim \lambda(z_1, z_2) \end{matrix}\right\}$$

Why $\mathcal{B}$?

① $G \rightsquigarrow \mathcal{B}$ proj variety

② Borel Subgroups are important for Rep theory of G

③ Borel-Weil uses $\mathcal{B}$

④ Beilinson-Bernstein uses $\mathcal{B}$

Let $G \curvearrowright V$ irreducible $\overset{\text{f.d.}}{\vee}$ rep'n

$B \leftrightarrow \ell_B \subset V$

$\mathcal{B} \leftrightarrow \{\ell_B \subset V\}_{B \in \mathcal{B}}$

$$\begin{tikzcd} B \curvearrowright \ell_B \text{ : 1-dim} \arrow[d, "ab"'] \\ B/[B,B] =: H \end{tikzcd}$$

$[B,B] = N$ nilpotent radical

$G = SL_n \rightsquigarrow H = (\mathbb{C}^\times)^{n-1}$

$H \curvearrowright \{\ell_B\}_{B \in \mathcal{B}}$ $\quad \chi : H \to \mathbb{C}^\times$ character

<u>Defn</u> A G-equivariant vector bundle $E \to Y$, $G \curvearrowright Y$ is a vector-bundle $E \to Y$ together with

$$\sigma^* E = \pi_2^* E$$

where $\underset{\text{action}}{\sigma : G \times Y \to Y}$ $\quad \pi_2 : G \times Y \to Y$ $\overset{\shortparallel}{\text{projection}}$

In particular $E_x \underset{\substack{\uparrow \\ \text{linear iso}}}{\overset{\sim}{=}} E_{g \cdot x}$ $\quad \forall x \in Y,\ g \in G$

G - equiv vector bundles on Y

$\Leftrightarrow$ vector bundles on $G \backslash Y$

$\{G$-equiv line bundles on $G/B\}$

$\Leftrightarrow$ $\{$line bundles on $pt/B\}$

$\Leftrightarrow$ $\{1$-dim rep'ns of $B\}$

$\Leftrightarrow$ $\{1$-dim rep'ns of $H\}$

$\Leftrightarrow$ $\{$characters of $H\}$

$$\chi : H \to \mathbb{C}^\times \rightsquigarrow \overset{\substack{G\text{-equivariant} \\ \Downarrow}}{\mathcal{L}_\chi} \text{ on } G/B$$

$$\mathcal{L}_{\chi,B} = \ell_B^*$$

G-equiv vector bundle E

For its section s

$(g \circ s)(x) = g s(g^{-1}x)$

$\Gamma(Y, E)$ is G-rep'n

<u>Thm</u> (Borel-Weil)

If λ is a dominant weight,

then $H^0(G, \mathcal{L}_\lambda)$ is rep'n with

Highest weight λ

<u>Fact</u> Any f.d. irrep of G is a highest weight.

$\Rightarrow$ Any such rep arises in this way

Ex $G = SL_2$, $\mathcal{B} = \mathbb{P}^1$

$\lambda = 0 \rightsquigarrow \mathcal{L}_0 = \mathbb{P}^1 \times \mathbb{C}$

$H^0(\mathcal{B}, \mathcal{L}_0) = \mathcal{O}(\mathbb{P}^1) = \mathbb{C}$

$\underset{n \geq 0}{\lambda = n} \rightsquigarrow \mathcal{L}_\lambda = \mathcal{O}(n)$

$H^0(\mathcal{B}, \mathcal{L}_\lambda) = H^0(\mathbb{P}^1, \mathcal{O}(n))$

deg n polys in x, y

Rmk For general λ,
one can describe $H^i(\mathcal{B}, \mathcal{L}_\lambda)$ ... due to Bott

(1) 2. Beilinson – Bernstein localization

rep'ns of $\mathfrak{g}$ of arbitrary, not f.d.

$H^0(\underset{\substack{\uparrow \\ \text{proj}}}{\mathcal{B}}, \mathcal{L}_\lambda) \to$ f.d. $\Rightarrow$ can't just look at line bundle

Slogan: "Rep theory of $G \subseteq$ Geometry of $\mathcal{B}$"

Let $G \curvearrowright X$ smooth$/\mathbb{C}$

$\rightsquigarrow \mathfrak{g} \to \mathrm{Vect}(X)$

$U(\mathfrak{g}) = \Gamma(X, \mathcal{D}_X) = \mathcal{D}(X)$

$[\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X = \mathcal{O}(X)]$

$\mathcal{O}_X \subset \mathcal{D}(X)$

$\mathcal{D}_X$ - mod $\xrightarrow{\Gamma} D_X(X)$ - mod $\xrightarrow{\varphi^*} U(\mathfrak{g})$ - mod

Take $X = \mathcal{B}$

Thm Belinson - Bernstein

$\Phi : \mathcal{U}(\mathfrak{g}) \to \mathcal{D}(\mathcal{B})$ is surjective

---

classical limit = associated graded

$$\begin{tikzcd}
\mathcal{U}(\mathfrak{g}) \arrow[rr] & & \mathcal{D}(\mathcal{B}) \arrow[d, "\mathrm{Gr}"] \\
\mathrm{Sym}(\mathfrak{g}) = \mathcal{O}(\mathfrak{g}^*) \arrow[u, squiggly] \arrow[rr] & & \mathcal{O}(T^*\mathcal{B}) = \mathcal{O}(\mathcal{N})
\end{tikzcd}$$

$\mathcal{N} \subset \mathfrak{g}^*$

$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ $a^2 + bc = 0$

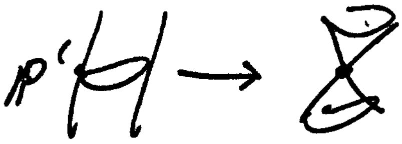
$$\begin{tikzcd}
T^*\mathcal{B} \arrow[rr, "\mu"] \arrow[dr, two heads] & & \mathfrak{g}^* \\
& \mathcal{N} \arrow[ur] &
\end{tikzcd}$$

Show $\mathcal{O}(\mathfrak{g}^*) \twoheadrightarrow \mathcal{O}(\mathcal{N})$ surj

and $\mathcal{N} \hookrightarrow \mathfrak{g}^* \Rightarrow$ done

Rmk $T^*\mathcal{B} \to \mathcal{N}$

Springer resolution



Study of $T^*\mathcal{B} \to \mathcal{N}$ or its variants

= Springer ~~resolution~~
theory

What is $\ker \Phi : \mathcal{U}(\mathfrak{g}) \to \mathcal{D}(\mathcal{B})$

Consider $Z(\mathfrak{g}) = Z(\mathcal{U}(\mathfrak{g}))$

By Schur's lemma, for irrep V of $\mathfrak{g}$

$Z(\mathfrak{g}) \circlearrowright V$ as a scalar

$z.v = \chi(z) \cdot v$ where $\chi : Z\mathfrak{g} \to \mathbb{C}$
central character

$Z\mathfrak{g} \simeq (\mathrm{Sym}\,\mathfrak{h})^W$ $\quad \mathfrak{h} \subset \mathfrak{g}$

Cartan

$\cong$

$\mathbb{C}[x_1, \dots, x_r]$

where $x_1 \dots x_r$ are W-symmetric polys

W weyl group

$r = \dim \mathfrak{h} = \mathrm{rk}\, \mathfrak{g}$

$\chi : Z\mathfrak{g} \to \mathbb{C}$

$\Rightarrow \chi \in \mathrm{Spec}(Z\mathfrak{g}) \cong \mathbb{A}^r$

Given rep'n of $\mathfrak{g}$

- first try to understand {irreps}
- try to understand how $Z\mathfrak{g}$ acts

Irrep $\mathfrak{g}$

/ $\mathrm{spec}(Z\mathfrak{g})$

$\Phi : U\mathfrak{g} \twoheadrightarrow \mathcal{D}(X)$

$\mathcal{D}(X)\text{-mod} \xrightarrow{\Phi^*} U(\mathfrak{g})\text{-mod}$

$\ker \Phi = U(\mathfrak{g}) \cdot \ker \chi_0$

$\Rightarrow \mathcal{D}(X)\text{-mod} \xrightarrow{\sim} U(\mathfrak{g})\text{-mod}_{\chi_0}$

where $Z\mathfrak{g}$ acts through χ_0

$\mathcal{D}^\lambda(X)\text{-mod} \xrightarrow{\sim} U\mathfrak{g}\text{-mod}_{\chi_\lambda}$

$\uparrow$

twisted diff. op.

$$\mathcal{D}_X\text{-mod} \xrightarrow{\Gamma} \mathcal{D}(X)\text{-mod} \xrightarrow[\sim]{\Phi^*} U(\mathfrak{g})\text{-mod}_{\chi_0}$$

$\uparrow$

when $X = \mathcal{B} = G/B$

X variety

$$\begin{tikzcd} QCoh(X) \arrow[r, shift left, "\Gamma"] & \mathcal{O}(X)\text{-mod} \arrow[l, shift left, "\Delta"] \end{tikzcd}$$

$\uparrow$

localization for $X = \operatorname{spec} A$

$$(\Delta M)(U_f) = M_f$$

$U_f = \{f \neq 0\}$

$$\begin{tikzcd} \mathcal{D}_X\text{-mod} \arrow[r, shift left, "\Gamma"] & D(X)\text{-mod} \arrow[l, shift left, "\Delta"] \end{tikzcd}$$

$$M \mapsto \Delta M = \mathcal{D} \otimes_{\mathcal{D}_X} M$$

<u>Thm</u> (BB)

$X = \mathcal{B} = G/B$ is <u>D-affine</u>

$$\begin{tikzcd} U\mathfrak{g}\text{-mod}_{\chi_0} \arrow[rr, "\Delta"] & & \mathcal{D}_{\mathcal{B}}\text{-mod} \arrow[dl, "\Gamma", "\wr"'] \\ & \mathcal{D}(\mathcal{B})\text{-mod} \arrow[ul, "\Phi^*", "\wr"'] & \end{tikzcd}$$

$M \in U(\mathfrak{g})$-mod

$$\Delta M = \mathcal{D} \otimes_{\mathcal{D}(X)} M$$

<u>Ex</u> $G = SL_2$, $\mathcal{B} = \mathbb{P}^1$

$U_1 = \{[z_1, z_2] : z_2 \neq 0\}$ $\quad z = z_1/z_2$ $\quad$ on $U_1 \cap U_2$

$U_2 = \{[z_1, z_2] : z_1 \neq 0\}$ $\quad x = z_2/z_1$ $\quad z = 1/x$

$$\bar{G} = \left\{ \begin{pmatrix} z_1 & * \\ z_2 & * \end{pmatrix} : \underset{\substack{\shortparallel \\ 0}}{(z_1, z_2)} \sim \lambda(z_1, z_2) \right\}$$

$$G \circlearrowright \bar{G} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \frac{a+bx}{c+dx}$$

$$\frac{d}{dz} = -x^2 \frac{d}{dx}$$

$$\mathrm{Vect}(\mathbb{P}^1) = H^0(\mathbb{P}^1, \mathcal{O}(2))$$

$$= \left\langle \frac{d}{dz}, z\frac{d}{dz}, z^2\frac{d}{dz} \right\rangle$$

$$\mathfrak{g} \to \mathrm{Vect}(X)$$

$sl_2 \to \mathrm{Vect}\,\mathbb{P}^1$ Lie alg. map

$$\left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)\varphi(z) = \left.\frac{d}{dt}\right|_{t=0} \varphi(e^{-t\xi}z)$$

$$\underset{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}{\overset{}{e}} \overset{}{,} \underset{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}{F}, \underset{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{h} \quad \Rightarrow \quad \begin{aligned} e &\to -\frac{d}{dz} \\ f &\mapsto z^2\frac{d}{dz} \\ h &\mapsto -2z\frac{d}{dz} \end{aligned}$$

$$\mathcal{U}(sl_2) \twoheadrightarrow \mathcal{D}(X)$$

$$Z = \mathbb{C}(c) \quad c = ef + fe + \frac{h^2}{2} \Rightarrow \mathcal{U}sl_2 / \mathcal{U}sl_2 \cdot c \xrightarrow{\sim} \mathcal{D}(X)$$

$$c \mapsto 0 \qquad \mathcal{D}_X\text{-mod} \rightsquigarrow \mathcal{U}sl_2\text{-mod}_{\chi_0}$$

$$\begin{tikzcd}
\mathbb{A}^1 & \mathcal{O}(\mathbb{A}^1) \arrow[d, equal] & \mathcal{O}(\mathbb{A}^1\setminus\{0\}) \arrow[d, equal] & \delta_0 \arrow[d, phantom, "\cup"] & \\
0 \arrow[r] & \mathbb{C}[x] \arrow[r] \arrow[d, equal] & \mathbb{C}[x,x^{-1}] \arrow[r] \arrow[d, equal] & \mathbb{C}[x,x^{-1}]/\mathbb{C}[x] \arrow[r] \arrow[d, equal] & 0 \\
 & D/D\partial \cong M_1 & D/D(\partial x) \cong M_{1/x} & D/Dx \cong M_{\delta_0} & \\
 & \{\partial\cdot 1=0\} \arrow[u, Rightarrow] & \{\partial x\cdot 1/x=0\} \arrow[u] & \{x\cdot\delta_{P_0}=0\} \arrow[u] &
\end{tikzcd}$$

D-mod Rep.

- $\mathcal{O}_{\mathbb{P}^1} \overset{\Gamma}{\rightsquigarrow} \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = \mathbb{C} = L_0$

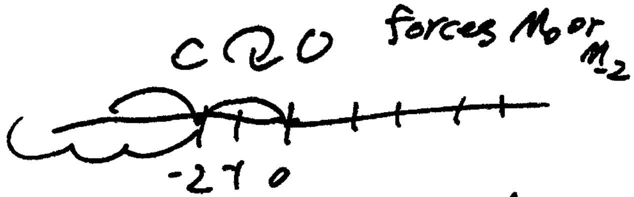
  simple module w/ h.w. 0

$\mathbb{A}^1 \overset{j}{\hookrightarrow} \mathbb{P}^1$

- $j_*\mathcal{O}_{\mathbb{A}^1 = \mathbb{P}^1\setminus\{\infty\}} \rightsquigarrow \Gamma(\mathbb{P}^1, j_*\mathcal{O}_{\mathbb{A}^1}) = \mathbb{C}[x] = \mathbb{C}[z^{-1}]$

  $= M_0^\vee$

  dual verma module w/ h.w. 0



- $j_!\mathcal{O}_{\mathbb{A}^1} \rightsquigarrow = M_0$

  $\mathrm{im}(M_0 \to M_0^\vee) = L_0$

  $\mathrm{im}(j_! \to j_*) = j_{!*}$

$L_0 = M_0/M_{-2}$

$L_0 \subset M_0^\vee$

$M_{-2} = M_0^\vee/L_0$

- $j_*\mathcal{O}_{\mathbb{A}^1}/\mathcal{O}_{\mathbb{P}^1}$ only around 0

$$\underset{\displaystyle j_*\mathcal{O}_{\mathbb{A}^1\setminus\{\infty\}}/\mathcal{O}_{\mathbb{P}^1\setminus\{\infty\}}}{\cup} = \delta_0 \rightsquigarrow \Gamma(\mathbb{P}^1, \delta_0) = \mathbb{C}[z^{-1}]/\mathbb{C}$$

$$= \underset{\displaystyle \| }{M_{-2}} = \underset{\displaystyle \| }{L_{-2}}$$

$$= M_{-2}^\vee = L_{-2}^\vee$$

$\mathfrak{g}$-rep ↔ (twisted) D-mod on $\mathcal{B}$

$(\mathfrak{g}, H)$-rep ↔ H-equivariant D-mod on $\mathcal{B}$

Lie $H \subset \mathfrak{g}$

Fun$(K\backslash H/K)$

Rmk (Fate) $H = B$

$$\begin{tikzcd}
(\mathfrak{g}, B)\text{-rep} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & B\text{-equiv D-mod on } \mathcal{B} \\
\text{h.w. theory (category } \mathcal{O}) \arrow[r, leftrightarrow] & B\backslash G/B \text{ Schubert cells}
\end{tikzcd}$$

Rmk $H = K \subset G$ compact

$\mathfrak{g} = \mathfrak{sl}_2$, $K = SO_2(\mathbb{C}) = \mathbb{C}^\times$

▨ ↔ $G_{\mathbb{R}}$ rep'n

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(2)

1. Fourier-Mckay transform for D-modules and Geometric Langlands correspondence for $G = GL_1$

A abelian variety → $A^\vee$ dual abelian variety

$QC(A) \simeq QC(A^\vee)$

deg 0 line bundle $\mathcal{L}$ on A ↔ $\mathcal{O}_{\mathcal{L}}$ skyscraper

$D(\mathrm{Bun}_G) \simeq ?$

$D(A) \simeq ?$

$$\begin{tikzcd} \{A \arrow[r] \arrow[d] & B\mathbb{G}_m\} \arrow[r, leftrightarrow] & A^\vee \\ QC(A) & & \end{tikzcd}$$

$A^\flat = \{\text{flat line bundles on } A\}$

<u>Thm</u> (Laumon, Rothstein)

$$\begin{tikzcd} D(A) \arrow[r, phantom, "="] & QC(A^\flat) \\ (\mathcal{L},\nabla) \text{ flat line bundle} \arrow[u] \arrow[r, leftrightarrow] & \mathcal{O}_{(\mathcal{L},\nabla)} \text{ skyscraper} \end{tikzcd}$$

~~let $A \cong \mathrm{Jac}\, C$~~

$$\begin{tikzcd} T^*A \arrow[r, phantom, "="] & A \times H^0(A,\Omega_A) \arrow[dr] & & A^\vee \times H^0(A,\Omega_A) \arrow[dl] \\ A \cong \mathrm{Alb}\, A \arrow[u] & & H^0(A,\Omega_*) & \end{tikzcd}$$

$$\begin{tikzcd} QC(A) \arrow[r, phantom, "\simeq"] \arrow[d, squiggly, "\text{def}"] & QC(A^\vee \times H^0(A,\Omega_A)) \arrow[d, squiggly, "\text{def}"] \\ D(A) \arrow[r, phantom, "\simeq"] & QC(A^\flat) \end{tikzcd}$$

let $A = \mathrm{Jac}\, C \Rightarrow A^\flat = \{\text{Flat line bundles on } C\}$

$= \underline{\mathrm{Pic}^0}\, C \qquad = \underline{\mathrm{Flat}_1}\, C$

$$\begin{array}{ccc} \rightsquigarrow D(\underline{\mathrm{Pic}^0} C) & \cong & QC(\underline{\mathrm{Flat}_1} C) \\ \cup & & \cup \\ ? & \xleftarrow[\text{loc}]{\cong} & ? \end{array}$$

$\underline{\text{Flat}}_1 C \xrightarrow{\pi} \text{Jac}\, C$

$\pi^{-1}(\mathcal{O}_C) \to \mathcal{O}_C$

$\mathcal{E} = (\mathcal{O}_C, d+\omega)$ skyscraper on $\underline{\text{Flat}}_1 C$

$\omega \in H^0(C, \Omega_C)$

$\rightsquigarrow (\mathcal{O}_{Jac}, d+\tilde{\omega})$ flat line bundle on Jac C

$\tilde{\omega} \in H_0(Jac, \Omega_{Jac})$

$\shortparallel$

$\mathcal{F}_{\mathcal{E}}$

$$H^0(C, \Omega_C) = H^0(Jac, \Omega_{Jac})$$

$$\omega \rightsquigarrow \tilde{\omega}$$

$$T\,Jac = Jac \times H^1(C, \mathcal{O}_C)$$

$$Vect(Jac) = H^1_0(C, \mathcal{O}_C)$$

$\partial_\xi = H^1(C, \mathcal{O}_C)$

$\nabla_{\partial_\xi} = \partial_\xi + \langle \omega, \partial_\xi \rangle$

flat connection on Jac

$$D_{Jac} = \mathcal{D}(Jac) = \text{Sym}\, H^1(C, \mathcal{O}_C)$$

$$= \mathcal{O}(H^0(C, \Omega_C))$$

$$\lambda_\omega : D_{Jac} \to C$$

$$\partial_\xi \mapsto -\langle \partial_\xi, \omega \rangle$$

$$\mathcal{F}_{\mathcal{E}} = \mathcal{D}/\ker \lambda_\omega = \mathcal{D} \otimes_D \mathbb{C}_\omega$$

Hitchin Fibration	$T^*Jac \to H^0(C,\Omega_C)$	$T^*Bun_G \to \mathcal{B}$
Hitchin section	$H^0(C,\Omega_C) \subset T^*Jac$	$\mathcal{B} \to HS \subset T^*Bun_G$
integrability	$\mathcal{O}(\mathcal{B}) = \mathcal{O}(T^*Jac)$	$\mathcal{O}(\mathcal{B}) = \mathcal{O}(HS)$ $= \mathcal{O}($
quantization	$\mathcal{O}(\mathcal{B}) = D_{Jac}$ $\parallel$ $U(H^1(C,\mathcal{O}_C))$	$\mathcal{O}(\wp_s) = H_\bullet(C,\mathfrak{g})$

	Jac	Bun_G
$U(\mathfrak{g}) \to \mathcal{D}_X$-mod $M \mapsto \mathcal{D} \otimes_{U\mathfrak{g}} M$	$M \mapsto D \otimes_{D_{Jac}} M$ $D_{Jac} = U(H^1(C,\mathcal{O}))$	$M \mapsto D \otimes_D M$

II Geometric Langlands Theory via Derived Algebraic Geometry

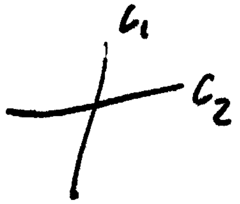
$\boxed{5}$ Introduction to DAG

everything $/k$ $\mathrm{char}(k) = 0$

<u>Thm</u>/ Bezout's Theorem $/ k = \bar{k}$

Consider $\mathbb{P}^2_k$ projective plane and $C_1, C_2 \subset \mathbb{P}^2_k$ smooth curves of deg m, n intersecting at a finite number of points, Then $mn = \sum_{x \in C_1 \cap C_2} \dim_k (\mathcal{O}_{C_1} \otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{O}_{C_2})_x$ holds

<u>Ex</u> Consider $\mathbb{A}^2_k$ curves $C_1 = \{x\}$ $C_2 = \{y\}$ of deg 1



$$\mathcal{O}_{C_1} \otimes_{\mathcal{O}_{\mathbb{A}^2}} \mathcal{O}_{C_2}$$
$$= k[x,y]/(x) \otimes k[x,y]/(y)$$
$$\cong k[x,y]/(x,y) \cong k$$

<u>Ex</u> $\mathbb{A}^2$ $C_1 = (y - x^2)$ $C_2 = (y = 0)$

$$k[x,y]/(y-x^2) \otimes_{k[x,y]} k[y,y]/(y)$$
$$\cong k[x]/(x^2) \quad \dim = 2$$

$$\mathcal{O}_{C_1} \otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{O}_{C_2} := \mathcal{O}_{C_1 \cap C_2}$$

Question: what happens when $C_1 = C_2$
most degenerate case

Ex $C_1 = (x) \subset \mathbb{P}^2$
$C_2 = (x) \subset \mathbb{P}^2$

$$k[x,y]/(x) \otimes_{k[x,y]} k[x,y]/(x)$$

$\uparrow$

one needs a resolution of this as $k[x,y]$ algebra

$$\overset{-1}{\varepsilon\, k[x,y]} \xrightarrow{d} \overset{0}{k[x,y]} \to k[x,y]/(x)$$

$\deg \varepsilon = -1$ $\quad \| \quad k[x,y,\varepsilon]$ $\qquad f \cdot g = (-1)^{|f||g|} g \cdot f$

$$\varepsilon \cdot \varepsilon = (-1)^{1\cdot 1} \varepsilon \cdot \varepsilon \Rightarrow \varepsilon^2 = 0$$

$$d(\varepsilon f) = d\varepsilon \cdot f + \underset{0}{\varepsilon \cancel{df}}$$

Commutative differential graded algebra,
CDGA

$C^\bullet = \bigoplus_d C^n[-n]$

$$\left(\varepsilon\, k[x,y] \xrightarrow{x\, d/d\varepsilon} k[x,y]\right) \otimes_{k[x,y]} k[x,y]/(x)$$

$$= \varepsilon\, k[x,y]/(x) \xrightarrow[\| \; 0]{x \cdot d/d\varepsilon} k[x,y]/(x)$$

$$= \varepsilon\, k[y] \xrightarrow{d=0} k[y] = k[y][1] \oplus k[y]$$

$C^\bullet$ cochain complex

$C^\bullet[n]$
$= C^{\bullet + n}$

$\mathcal{O}_{\mathbb{P}^1} \otimes_{\mathcal{O}(\mathbb{P}^2)} \mathcal{O}_{\mathbb{P}^1}$

$$0 \to \underset{\substack{\mathbb{P}2 \\ \mathcal{O}_{\mathbb{P}^2}(-\{x=0\})}}{\mathcal{O}(-1)} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathcal{O}_{\mathbb{P}^1}} \to 0$$

$\mathcal{O}_{\mathbb{P}^1}(-1)[1] \otimes \mathcal{O}_{\mathbb{P}^1}$

$\chi(\mathcal{O}_{\mathbb{P}^1}(-1)[1] \otimes \mathcal{O}_{\mathbb{P}^1}) = 1$

<u>Note</u> Grothendieck distinguished

$f=0$ and $f^2=0$.

DAG distinguishes

$f=0$ and $(f=0)^2$

We are led to $CDGA^{\leq 0}$ ~~instead~~ instead of Ring

$\boxed{Sch}$

<u>Defn</u>/ A derived scheme is a top'l space X with sheaf $\mathcal{O}_X$ valued in $CDGA^{\leq 0}$

s.t. ① ~~a~~ $t_0 = (X, H^0(\mathcal{O}_X))$ is a scheme

② $H^i(\mathcal{O}_X)$ is a quasicoherent sheaf over $t_0(X)$

$\forall i \in \mathbb{Z}$

(0 in degree positive)

Ex ① A sheme $(X, \mathcal{O}_X)$ is a derived scheme

② $A \in CDGA^{\leq 0}$ defines a derived scheme

$$(\operatorname{Spec} H^0 A, A)$$
$\uparrow$
affine derived scheme

$$cl\,sch^{aff} \leftrightarrows Ring$$
$$sch^{aff} \longleftrightarrow CDGA^{\leq 0}$$

Rmk

derived scheme: classical scheme
= classical scheme: reduced scheme

Sch is an ∞-category!

For a usual category $\mathcal{C}$, For $X, Y \in \mathcal{C}$

$\operatorname{Hom}_{\mathcal{C}}(X, Y)$ is a set

$\uparrow \pi_0$

$\operatorname{Map}_{\mathcal{C}}(X, Y)$ is a space

X cl. scheme

$$\underset{\text{Yoneda}}{\rightsquigarrow} h_X : (cl\,Sch^{aff})^{op} \to Set$$
$$S \longmapsto \operatorname{Hom}(S, X)$$

$$\tilde{h}_X : (Sch^{aff})^{op} \to Spc$$
$$S \longmapsto \operatorname{Map}(S, X)$$

$X_1 = \operatorname{Spec} A_1 \qquad X_2 = \operatorname{Spec} A_2$

$Y = \operatorname{Spec} B$

$X_1 \times_Y X_2$ fiber product

$\Leftrightarrow \quad A_1 \otimes^L_B A_2$

Expect $h_{X_1 \times_Y X_2}(S) \simeq h_{X_1}(S) \times_{h_Y(S)} h_{X_2}(S)$

Homotopy equivalence

not true!

true w/ $\tilde{h}_X$!

Everything is derived

$\text{Vect}_k :=$ cochain cpx

com Alg $:=$ CDGA $=$ com Alg(Vect)

QC ⊗ DG category

QCoh abelian category

$\mathcal{D}$ DG category

$\mathcal{D}_X$-mod abelian category

---

derived scheme is a Functor

$(\text{Sch}^{\text{aff}})^{op} \to \text{Spc}$

consider all such functors!

Prestacks! PreStk

These are the most general class of spaces that appear in alg geo.

(so far?)

Ex · (Betti stack)

M top'l space, $M \in Spc$

$M_B : (Sch^{aff})^{op} \to Spc$

$S \to M$ "constant function"

· (De-Rham stack)

$\mathcal{Y}$ prestack $\mathcal{Y}_{IR} : (Sch^{aff})^{op} \to Spc$

$S \mapsto \mathcal{Y}(S^{red})$

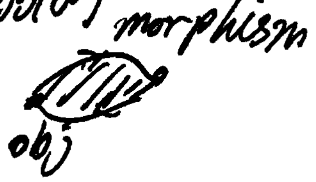
· (classifying stack)

$BG : (Sch^{aff})^{op} \to Spc$

$S \longmapsto G$-bundles on S

(which is groupoid)

$$\begin{tikzcd} Spc \arrow[d, phantom, "\shortparallel"] \arrow[d, leftrightarrow, shift left=4] \\ \infty\text{-grd} \end{tikzcd}$$



· (Mapping stack)

$\mathcal{X}, \mathcal{Y}$ prestacks

$\underline{Map}(\mathcal{X},\mathcal{Y})(S) = \underline{Map}(S \times \mathcal{X}, \mathcal{Y})$

can show $\underline{Map}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z}) = \underline{Map}(\mathcal{X}, \underline{Map}(\mathcal{Y}, \mathcal{Z}))$

Main Example

- X classical scheme

$$\mathrm{Map}(X, BG) =: \mathrm{Bun}_G(X)$$

$$\mathrm{Bun}_G(X)(S) = \mathrm{Map}(S \times X, BG)$$

G-bundles on $S \times X$

- $\mathrm{Map}(X_{dR}, BG) =: \mathrm{Flat}_G(X) =$ de-Rham moduli space of flat G-bundles on X
- M top'l space

$$\underline{\mathrm{Map}}(M_B, BG) =: \mathrm{Loc}_G(M)$$

$=$ character stack

$=$ Betti moduli

2) Quasi-coherent Sheaves

We want DG-category of quasi-coh sheaves on a pre-stack

<u>Defn</u> A DG category is a category enriched over Vect_k = cochain complexes $C_1, C_2 \in \mathcal{C}$ $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ is a complex.

<u>Ex</u>

- Vect dg cat

$\underline{\mathrm{Hom}}^{\bullet}_{\mathrm{vect}}(C^{\bullet}_{d_C}, D^{\bullet}_{d_D})$ is a cochain complex

$$\begin{cases} \mathrm{Hom}^k_{\mathrm{vect}}(C^{\bullet}, D^{\bullet}) = \prod \mathrm{Hom}(C^i, D^{i+k}) \\ D((\varphi^k)_{k\in\mathbb{Z}}) = (d_D \circ \varphi^k - (-1)^{k}\varphi^{k} d_C)_{k\in\mathbb{Z}} \end{cases}$$

$$\begin{tikzcd} C^0 \arrow[r, "d_C"] \arrow[d, "\varphi^0"] & C^1 \arrow[r, "d_C"] \arrow[d, "\varphi^1"] & C^2 \arrow[d, "\varphi^2"] \\ D \arrow[r, "d_D"'] & D^1 \arrow[r, "d_D"'] & D^2 \end{tikzcd}$$

- DG alg A
  is a DG cat w/ single obj
  $End(\cdot) = A$

- DG alg A
  A-mod . DG cat of DG modules
  $M = \bigoplus M^n$
  $A^i \cdot M^j \subset M^{i+j}$ $\quad d_M(a \cdot m) = d_A a \cdot m + (-1)^{(a)} a \cdot d_M$

default assumption on DG categories

$\boxed{DGCat_{cont.}}$
- co-complete: has all colimits
- pre-triangulated: $Ho(\mathcal{C})$ is triangulated
  $\uparrow$
  obj. are the same
  morphisms $= H^0(\dots)$

- Functors are continuous
  btwn DG-cat = preserves colimits

$A \xrightarrow{f} B$
$\text{spec } B \to \text{spec } A$
f_*: B-mod $\to$ A-mod
f^*: A-mod $\to$ B-mod $\otimes_B$
$\quad \otimes_A$
$M \mapsto f^*M = B \otimes_A M$

A-mod-mod Functor

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "id \otimes a."] \arrow[d, "a \otimes id."'] & A \otimes M \arrow[d, "a."] \\
A \otimes M \arrow[r, "a."] & M
\end{tikzcd}$$

Exer

$$\begin{tikzcd}
A\text{-mod} \otimes A\text{-mod} \otimes B\text{-mod} \arrow[r] \arrow[d] & {} \arrow[d] \\
{} \arrow[r] & {}
\end{tikzcd}$$

$\Leftrightarrow$ projection formula

$\mathcal{Y}$ -prestack $\qquad S = \operatorname{Spec} A$

$\qquad\qquad\qquad QC(S) = A\text{-mod}$

$$QC(\mathcal{Y}) := \lim_{\begin{pmatrix} S \xrightarrow{y} \mathcal{Y} \\ S \in Sch^{aff} \end{pmatrix}^{op}} QC(S) \quad \text{in } DGCat_{cont}.$$

$\Downarrow$

$\mathcal{F} \qquad (S, y) \rightsquigarrow \mathcal{F}_{S,y} \in QC(S)$

$S' \xrightarrow{f} S \xrightarrow{y} \mathcal{Y} \rightsquigarrow \mathcal{F}_{S',y} \simeq f^*\mathcal{F}_{S,y}$

A

Rmk 1. This defn. is different from the usual defn. even for a classical scheme! When they are comparable, they coincide.

Formal Properties 1

- $\mathcal{Y} = \operatorname{Spec} A \rightsquigarrow QC(\cancel{\mathcal{Y}}) = A\text{-mod}$
- $\mathcal{O}_{\mathcal{Y}} \Leftrightarrow \{\mathcal{O}_S \in QC(S)\}$

$f^*: A\text{-mod} \to B\text{-mod}$

$M \mapsto B \otimes_A M$

$A \mapsto B$

$QC(\mathcal{Y})$ is a symm monoidal category w/ $\mathcal{O}_{\mathcal{Y}}$ as unit.

$\mathcal{X} \xrightarrow{f} \mathcal{Y}$ map of prestacks

$$f^*: QC(\mathcal{Y}) \to QC(\mathcal{X})$$
$$\mathcal{F} \to f^*\mathcal{F}$$

$$\begin{tikzcd} S \arrow[r, "x"] \arrow[dr, "f\circ x"'] & \mathcal{X} \arrow[d, "f"] \\ & \mathcal{Y} \end{tikzcd} \qquad \mathcal{F}_{S, f\circ x} = (f^*\mathcal{F})_{S,x}$$

$$QC^*: PreStk \to DGCat_{cont.}$$
$$\mathcal{Y} \to QC(\mathcal{Y})$$
$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \to f^*: QC(\mathcal{Y}) \to QC(\mathcal{X})$$

How about F_*

Thm (Adjoint Functor thm.)
① Any cont. Functor admits right adj. (cont.)
② Any functor preserving limits admits left adjoint.

$$\begin{aligned} Hom(F(colim\, X_i), Y) &= Hom(colim\, X_i, GY) \\ &= lim\ Hom(X_i, GY) \\ &= lim\ Hom(F(X_i), Y) \\ &= Hom(colim\, F(X_i), Y) \end{aligned}$$

f^* cont. $\rightsquigarrow$ $F_*: QC(\mathcal{X}) \to QC(\mathcal{Y})$ not cont. in general

For

$$\begin{tikzcd}
\mathcal{X}_1 \arrow[r, "g'"] \arrow[d, "f'"'] & \mathcal{Y}_1 \arrow[d, "f"] \\
\mathcal{X}_2 \arrow[r, "g"'] & \mathcal{Y}_2
\end{tikzcd}$$

$QC(\mathrm{Flat}_G)$

$\exists\ g^* \circ f_* \to f'_* \circ g^*$ base change morphism

adj $\quad id \to g'_* \circ g'^*$

$$\begin{tikzcd}
f_* \arrow[r] \arrow[dr] & f_* g'_* \circ g'^* \arrow[d, equal] \\
 & g_* f'_* g'^*
\end{tikzcd}$$

use adj $\to \boxed{g^* f_* \to f'_* g'^*}$

How about D-modules?

$\mathcal{Y}$ prestack

$\mathcal{Y}_{dR}$ de-Rham stack (recall D-mod is QC + Flat connection)

$$QC(\mathcal{Y}_{dR}) =: D^!(\mathcal{Y})$$

Rmk. $\underset{X}{\mathcal{Y} =}$ classical smooth scheme

$D(X)$ is equal to DG cat of (left) D_X-modules

$D^{\dagger,!} \quad \mathrm{PreStk} \longrightarrow \mathrm{DGCat}_{cont.}$

$\quad \|$

$QC^* \circ (-)_{dR}$

$f_* \rightsquigarrow f_{*,dR} : D(\mathcal{X}) \to D(\mathcal{Y})$
de-Rham pushforward

$\dfrac{\mathcal{X} \to \mathcal{Y}}{\mathcal{X} \to \mathcal{X}_{dR}}$

$p^* : D(\mathcal{X}) \xrightarrow{obl.} QC(\mathcal{X})$ "oblivion functor"

$\mathcal{X} = X$ classical
this comes from $\mathcal{O}_X \to \mathcal{D}_X$

$$\begin{tikzcd}
D(\mathcal{Y}) \arrow[r, "obl"] \arrow[d, "f^\dagger"'] \arrow[dr, phantom, "\circlearrowright"] & QC(\mathcal{Y}) \arrow[d, "f^*"] \\
D(\mathcal{X}) \arrow[r, "obl"'] & QC(\mathcal{X})
\end{tikzcd}$$

but $f_{*,dR}$ doesn't give

$\mathrm{Spec}(A)\,(k[\varepsilon]/(\varepsilon^2)) = T[\mathrm{Spec}(A)]$ $\quad f_*$ in QC.

$\mathrm{Spec}(A)_{dR}(k[\varepsilon]/(\varepsilon^2)) = \mathrm{Spec}(A)(k)$

$\boxed{6}$ Back to Basics

Debts

- ∞ - category
- DG - category
- derived stacks
- BG, Bun_G
- $f_*: QC(X) \to QC(Y)$ continuous if f schematic, quasi-cont
- D-modules + De Rham Stack

1) Homotopical Algebra & ∞-categories

algebra	main obj	via	"in practice" res
homological	abelian cat A	$ch(A)$	proj. res (inj res)
homotopical	cat $\mathcal{C}$	$s\mathcal{C}$ simplicial obj in $\mathcal{C}$	cofibrant res (Fibrant res)

- $A \hookrightarrow \underset{\text{chains?}}{ch(A)}$ in deg 0
- $\mathcal{C} \hookrightarrow s\mathcal{C}$ a const. object
- If we regard A as $\mathcal{C}$, then compatibility is given by Dold-Kan correspondence
- D-K is equivalence of model sets
- D-K is equivalence of ∞-cats

Introduce a category Δ, <u>simplex</u> <u>category</u>

<u>obj</u>: $[n] = \{0, \dots, n\}$ $\quad n \in \mathbb{Z}_{\geq 0}$

<u>mor</u>: monotonic, non-decreasing maps

$d^i : [n-1] \to [n]$ "misses" i, $\quad 0 \leq i \leq n$

coface map $\{0, \dots, n\} \to \{0, \dots i-1, i+1, \dots n\}$

$s^j : [n] \to [n-1]$ $\qquad 0 \leq j \leq n-1$

codegeneracy map $\{0, \dots, n\} \to \{0, \dots i, i, \dots n-1\}$

<u>Defn</u>| A simplicial set is a functor

$$X : \Delta^{op} \to \text{set}$$

$$[n] \to X([n]) = X_n$$

$$d^i \to d_i = X(d^i) : X_n \to X_{n-1}$$

$$s^j \to s_j = X(s^j) : X_{n-1} \to X_n$$

$$\begin{tikzcd}
X_0 \arrow[r, "s_0^0" description] & X_1 \arrow[l, bend right, "d_0^0"'] \arrow[l, bend left, "d_1^0"] \arrow[r, bend left=20, "s_0"] \arrow[r, bend right=20, "s_1"'] & X_2 \arrow[l, bend right=50, "d_0"'] \arrow[l, "d_1" description] \arrow[l, bend left=50, "d_2"] & \cdots \\
\{v_0 \cdots v_l\} \arrow[u, equal] & \{e_1 \cdots e_m\} \arrow[u, equal] & \{f_1 \cdots f_n\} \arrow[u, equal] &
\end{tikzcd}$$

<u>claim</u>| This $^\uparrow$ diagram encodes the data of X_j.

Any morphism ~~of~~ is a composition of d_i, s_j

~~Still there are~~

~~Claim~~ ~~This~~

Still, there are relations:

e.g. $d^i d^{i+1} = d^i d^i$

(check) $\{0,1,\dots\} \to \{0,1,\dots i-1, i+2, \dots\}$

Exer (Simplicial identities)

There are more than these. Find them all.

To find the shape of a simplicial set X we define $|\cdot| : Set_\Delta \to Top$ "geometric realization"

$$X \to \coprod_{n \in \mathbb{Z}_{\geq 0}} \underset{\sim}{\underline{X_n \times |\Delta^n|}}$$

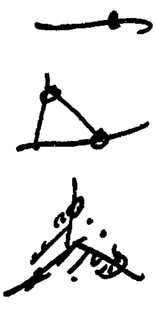
n-simplex

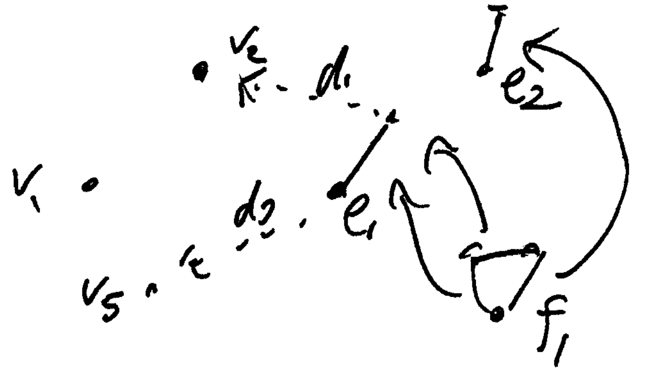
$|\Delta^n| = \{ t_0 \dots t_n \in \mathbb{R}^{n+1} : t_i \geq 0, \sum t_i = 1 \}$

n=0

n=1

n=2





Question: Is there a simplicial set Δ^n s.t. $|\cdot|(\Delta^n) = |\Delta^n|$

Answer: $\Delta^n = \mathrm{Hom}_\Delta(-,[n])$ Yoneda Functor

$X_n = \mathrm{Hom}(\Delta^n, X)$

$\mathrm{Sing} : \mathrm{Top} \to \mathrm{Set}$

$Y \to \mathrm{Sing}\, Y_n = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, Y)$

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, shift left, "|\cdot|"] & \mathrm{Top} \arrow[l, shift left, "\mathrm{sing}"] \end{tikzcd}$$

<u>convention:</u>

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

F is left-adjoint
G is right-adjoint

$$\begin{tikzcd} \mathcal{D} \arrow[d, bend left, "G"] \\ \mathcal{C} \arrow[u, bend left, "F"] \end{tikzcd}$$

sometimes

<u>Defn</u>/ A simplicial object in $\mathcal{C}$ is a functor $\Delta^{op} \to \mathcal{C}$

Ex: $\mathcal{C} = \mathrm{Set} \rightsquigarrow \mathrm{sSet}$
$\mathcal{C} = \mathrm{Ab} \rightsquigarrow \mathrm{sAb}$ simplicial abelian groups
$\mathcal{C} = \mathrm{Ring} \rightsquigarrow \mathrm{sRing}$ simplicial Rings

Dold Kan

Fix $\mathcal{A}$ abelian cat. (e.g. $\mathcal{A} = \mathrm{Ab}$)
consider $s\mathcal{A} \ni A$ $\quad A_n$ is abelian gp $\forall n \in \mathbb{Z}_{n \geq 0}$

<u>Defn</u>/ A Moore (unnormalized) chain complex associated to A is $C(A)$ where

$$\begin{cases} C(A)_n = A_n \\ d|_{A_n} = d_0 - d_1 + \dots + (-1)^n d_n \end{cases}$$

<u>Exer:</u> $d^2 = 0$

Defn/ A normalized chain complex $N(A)$ is defined as:

$$\begin{cases} N(A)_n = \bigcap_{i=1}^{n} \ker(d_i : A_n \to A_{n-1}) \\ d = d_0 \end{cases} \qquad N(A) \subset C(A) \text{ subcomplex}$$

$ch_{\geq 0} A = \{$ chain complexes in A concentrated in non-negative degrees

$(\cdots \to C_2 \to C_1 \to C_0 \to 0)$

Thm/ Dold-Kan

$\exists$ equivalence $sA \underset{\sim}{\overset{N}{\rightleftarrows}} ch_{\geq}(A)$

s.t. $\pi_n(A) \cong H_n(NA)$

"Idea for homotopy $\cong$ homology"

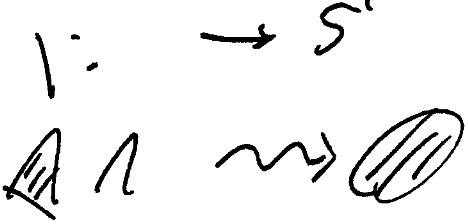
$\Delta^n/\partial\Delta^n \to A$ $\qquad \mathrm{Hom}_{sSet}(\Delta^n, A) = A_n$

$\sim S^n \to A$ $\quad \cdots\to \quad \mathrm{Hom}_{sSet}(\Delta^n/\partial\Delta^n, A)$

$= \bigcap_{i=0}^{n} \ker(d_i : A_n \to A_{n-1})$

$$\begin{tikzcd} \Delta^n/\partial\Delta^n \arrow[r] \arrow[d, "d^0"'] & A \\ \Delta^{n+1}/\Lambda_0^{n+1} \arrow[ur] & \end{tikzcd} \longleftrightarrow \text{null-homotopic maps}$$

$n=1$



Model Categories

<u>Idea</u> / Given a category $\mathcal{C}$, sometimes one might want to regard some morphisms as if they were iso, Model categories are supposed to help us!

Top $\ni X, Y$

$f: X \to Y$ is called a weak homotopy equivalence if $\pi_n f: \pi_n X \to \pi_n Y$ agrees $\forall n$

Ho(Top) obj: top'l spaces
mor: cont. maps
<u>Forcing</u> weak homotopy equivalence as homotopy equivalence

There is a nice class of spaces:

<u>Thm</u> / (Whitehead)
If X, Y are CW then $f: X \to Y$ weak homotopy e
is a homotopy equi

<u>Thm</u> / (CW approximation)
For $X \in$ Top, $\exists$CW complex QX s.t. $QX \xrightarrow{\sim} X$ by weak homotopy

---

Ho(Top)

||

CW complexes, homotopy equivalence.

$$\mathrm{Hom}_{Ho(Top)}(X, Y)$$

$$\wr\!\downarrow$$

$$\mathrm{Hom}_{Top}(QX, QY)$$

Take $\mathcal{C}$, with W the weak equivalences

$\mathcal{C}[W^{-1}] = Ho(\mathcal{C})$

In model category theory, we find classes of morphisms called

- W - weak equivalences
- fibrations
- cofibrations

They satisfy axioms:

category	Nice class of spaces	Approximation
Top	CW complexes	CW approx
CW	cofibrant-fibrant objects	$QX \xrightarrow{\sim} X$
sSet	Kan complexes	$QX \xrightarrow{\sim} X$
ch ≥0 ch ≤0	inj. modules proj. modules	$QX \xrightarrow{\sim} X$

like co-ordinate charts

2) DG categories

$\mathrm{DGCat}_{\mathrm{cont.}}$ presentable stable (cocomplete) having all limits

{ dg categories

{ Functors are continuous, preserving colimits

$\mathcal{C}_1, \mathcal{C}_2 \in$ DG Categories

want $\mathcal{C}_1 \otimes \mathcal{C}_2$

is a DG category enriched over Vect

Cat	Vect	$\mathrm{DGCat}_{\mathrm{cont}}$
	linear	cocomplete
	$V_1 \times V_2 \to W$ bilinear $\Leftrightarrow V_1 \otimes V_2 \to W$	$\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ bi-continuity $\mathcal{C}_1 \otimes \mathcal{C}_2 \xrightarrow{\text{cont.}} \mathcal{D}$
Unit	k	vect

$\Uparrow$

vector spaces with chain complex structurre!!

producti

$$\begin{tikzcd} & W \arrow[d, dotted] \arrow[ddl, bend right] \arrow[dr, bend left] & \\ & c_1 \otimes c_2 \arrow[dl] \arrow[r] & c_2 \\ c_1 & & \end{tikzcd}$$

Defn | $\mathcal{C}$ complete$_*$ DG category

- An object $c \in \mathcal{C}$ is called compact if $\mathrm{Hom}_{\mathcal{C}}(c,-) : \mathcal{C} \to \mathrm{Vect}$ is continuous

∘ $\mathcal{C}$ is called compactly ~~continuous~~ generated if

∃ compact objects c_α generating $\mathcal{C}$

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(c_\alpha, C) = 0 \qquad \forall \alpha \in I$$

nternal hom

$$\Rightarrow C = 0$$

Ex | $\mathrm{Vect}^{\heartsuit}$ the abelian cat of vector spaces

For which V is $\mathrm{Hom}_{\mathrm{Vect}^{\heartsuit}}(V,-)$ continuous?

$$V = \operatorname*{colim}_{i \in I} V_i$$

$$\underline{\mathrm{Hom}}(\mathrm{colim}\, X_i, Y) = \lim \underline{\mathrm{Hom}}(X_i, Y)$$

$$\underline{\mathrm{Hom}}(\lim X_i, Y) \longleftarrow \mathrm{colim}\, \underline{\mathrm{Hom}}(X_i, Y)$$

$$\underline{\mathrm{Hom}}(X, \mathrm{colim}\, Y_i) \Longleftarrow \mathrm{colim}\, \underline{\mathrm{Hom}}(X, Y_i)$$

$$\underline{\mathrm{Hom}}(X, \lim Y_i) = \lim \underline{\mathrm{Hom}}(X, Y_i)$$

V is ~~[illegible]~~ compact $\Leftrightarrow$ V is f.d.

$\Rightarrow$ colim $\underline{Hom}(V_i, V_i) \xrightarrow{\sim} \underline{Hom}(V, V)$

$f_i : V \to V_i \xrightarrow{\sim} id$

$i : V_i \to V$

$i \circ f = id_V \Rightarrow \dim V < \infty$ } think about this!

$\Leftrightarrow Hom(V, -) = V^* \otimes$ left adj. ■

$\mathcal{C}$ cocomplete DG

$\mathcal{C}^c$ full subcat of compact objects

<u>claim</u>/ $\mathcal{C}^c$ knows almost everything about $\mathcal{C}$

$\mathcal{C}^0$ small category $\rightsquigarrow$ Ind$(\mathcal{C}^0)$ ind-completi

s.t. Funct$(\mathcal{C}^0, \mathcal{D}) \simeq$ Funct$_{cont.}($Ind$(\mathcal{C}^0), \mathcal{D})$

<u>Thm</u>/ $\mathcal{C}^c \to \mathcal{C}$ full

Ind$(\mathcal{C}^c) \xrightarrow{\sim} \mathcal{C}$ is an equivalence

$\Leftrightarrow$ $\mathcal{C}$ is compactly generated

$S =$ Spec A affine derived scheme

A is a derived ring

---

scheme : Ring $\to$ Set

Prestack : Derived Ring $\to$ Derived Set

sAb $\simeq Ch_{\geq 0} \simeq Ch$

derived $\rightsquigarrow$ simplici

sRing $\simeq cdga^{\leq 0}$

if $\mathcal{C} = QC(S) = A\text{-mod}$

Claim $\mathcal{F} \in QC(S)$ is compact

$\Leftrightarrow$ $\mathcal{F}$ is perfect
$\mathcal{F} \in Perf(S)$ } what does this mean?

Defn $S = Spec\, A$ is almost of finite type if A is
is $H^0(A)$ is of finite type $/k$
$H^i(A)$ is of finite type $/ H^0(A)$

If S-classical
$A \in Ring$

A perfect complex is a finite complex of vector bundles on S.

In general perfect complex includes $\mathcal{O}_S$ and is closed under finite limits, colimits, sums.

$A = k[\epsilon]/(\epsilon^2)$
k is A-module

$\cdots A \xrightarrow{\cdot\epsilon} A \xrightarrow{\cdot\epsilon} A] \to k$

k is not perf complex
$k \in QC(S)$
but $k \notin Perf(S)$

Defn $\mathcal{F} \in Coh(S) \subset QC(S)$

$\Leftrightarrow$ $\mathcal{F}$ is cohomologically bounded and each cohomology is coherent over (finitely presented) $H^0(A)$

Ex (again) $A = k[\epsilon]/(\epsilon^2)$

$k \in Coh(S) \subset QC(S)$

$k \notin Perf(S)$

$Perf(S) \not\subset Coh(S)$

$\rotatebox{90}{$\notin$}$

$\mathcal{O}_S$

$= S \in cdga^{\le 0}$

$A = k[u]$ $\deg u = -2$

<u>Defn</u>| $S = Spec\ A$ is <u>eventually</u> <u>coconnective</u> if $H^i(A) = 0$ for $i \ll 0$

If S is eventually coconnective then $Perf(S) \hookrightarrow Coh(S)$

<u>Defn</u>| S is of affine type if it's of almost finite type & eventually coconnective

<u>Defn</u>| Let S be of affine type

$I.C. := Ind(Coh(S))$ $Coh(S) \to QC(S)$

$$\begin{tikzcd} QC(S) \arrow[r, shift left, "\Xi_S"] & IC(S) \arrow[l, shift left, "\Psi_S"] \end{tikzcd}$$

$QC(S) = Ind(Perf(S))$

If $\mathcal{C}$ compactly generated, the converse holds

<u>Lemma</u>|

- F is continuous
- If G is continuous, F sends cpt. to cpt.

Why I.C., not QC?

$f: X \to Y$

$f_*: QC(X) \to QC(Y)$

If f is proper then one expects its right adjoint is $\underset{\sim}{f}^!$

$\mathrm{Perf} \xrightarrow{f_*} \mathrm{Perf}$ <u>False</u>

$\mathrm{Coh} \xrightarrow{f_*} \mathrm{Coh}$ true

$\rightsquigarrow f^!$ is the natural functor to consider.

$\boxed{7}$ Singular support of Coherent sheaves

$$\begin{tikzcd} D(\mathrm{Bun}_G\Sigma) \cong QC(\mathrm{Flat}_{G^\vee}\Sigma) \arrow[d] & \mathrm{Perf} \arrow[dd, leftrightarrow] \\ \square \arrow[d] & \\ D(\mathrm{Bun}_G\Sigma) \cong IC(\mathrm{Flat}_{G^\vee}\Sigma) & \mathrm{Coh} \end{tikzcd}$$

is too naive

is also too naive

---

1) Tangent complex

X space $\rightsquigarrow$ T_X tangent bundle

X alg var. $\rightsquigarrow$ T_X tangent sheaf $\in QC(X)^\heartsuit$

if X not smooth $\neq$

X alg var. $\rightsquigarrow$ $\mathbb{T}_X$ tangent complex

As always, begin with affine scheme

$S := \mathrm{Spec}\, A \in \mathrm{Sch}^{\mathrm{aff}}$ or $A \in \mathrm{ComAlg}^{\leq 0}$

<u>Defn</u>| In steps:

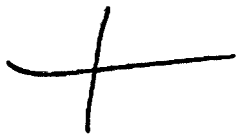
① $\mathrm{Der}^i A = \{\varphi : A \to A[i] \mid \varphi(fg) = \varphi(f)g + (-1)^{i|f|} f\varphi(g)\}$

② $\mathrm{Der}^\bullet A$

where $(d\varphi)(a) = d_A\varphi(a) + (-1)^{|\varphi|}\varphi(d_A a)$

③ $\mathbb{T}_S := \mathbb{T}_A := A \otimes_{\tilde{A}} \mathrm{Der}\,\tilde{A}$ where $\tilde{A}$ is quasi-free resolution

Ex| $A = k[x,y]/(xy)$



$\tilde{A} = k[x,y,\epsilon]$ $d\epsilon = xy\frac{\partial}{\partial \epsilon}$

is a $\overset{\text{u.f.}}{\smile}$ resolution of A

$\tilde{A} = \quad \varepsilon k[x,y] \to k[x,y]$

$\epsilon \to xy$

rule $f \cdot g = (-1)^{|f||g|} g \cdot f$

$f = g = \epsilon \Rightarrow \varepsilon^2 = (-1)^{1 \cdot 1} \varepsilon^2 \Rightarrow \varepsilon^2 = 0$

Facts|

- $\mathbb{T}_A$ is indep. of choice of $\tilde{A}$
- $\mathbb{T}_A$ is free A-module

eg. $A = k[x, \xi, \eta]$ $|x| = 0$, $|\xi| = -2$, $|\eta| = -5$

$\mathbb{T}_A$ is of rk 1 in degrees 0, 2, 5

$\frac{\partial}{\partial x} \ \frac{\partial}{\partial \xi} \ \frac{\partial}{\partial \eta}$

$$\mathbb{T}_A = (\underset{\frac{\partial}{\partial x}\ \ \frac{\partial}{\partial y}}{\overset{0}{\tilde{A} \oplus \tilde{A}}} \to \underset{\frac{\partial}{\partial \epsilon}}{\overset{1}{\tilde{A}}})$$

$\xi \to [d_A, \xi]$

$$\frac{\partial}{\partial x} \to xy\frac{\partial}{\partial \epsilon}\frac{\partial}{\partial x} - \frac{\partial}{\partial x} xy \frac{\partial}{\partial \epsilon} = -y\frac{\partial}{\partial \epsilon}$$

$$\frac{\partial}{\partial y} \to -x\frac{\partial}{\partial \epsilon}$$

① $s = (x, y) \neq (0,0)$

$\dim H^0(\mathbb{T}_{s,s}) = 1$

② $0 = (0,0)$

$\dim H^0(\mathbb{T}_{s,0}) = 2$

$\dim H^1(\mathbb{T}_{s,0}) = 1$

$\Rightarrow \chi(\mathbb{T}_{s,s}) = 1$

$\mathbb{H}_S = \mathbb{T}_S^\vee = \underline{Hom}_{QC(S)}(\mathbb{T}_S, \mathcal{O}_S)$

cotangent cpx

$H^i(\mathbb{T}_{s,s}[-1]) = H^{i-1}(\mathbb{T}_{s,s})$

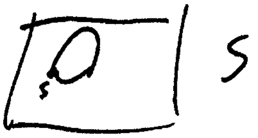
Rmk] (shifted tangent cpx)

$\mathbb{T}_{s,s}[-1]$ has lie alg. str. in Vect

$\Omega_s S \to pt$ based loop space $\Rightarrow$ group object

$$\begin{tikzcd}
\Omega_s S \arrow[r] \arrow[d] & pt \arrow[d] \\
pt \arrow[r] & S
\end{tikzcd}$$

"Fiber product"



In DAG, gp ⇝ Lie alg.

$Lie(\Omega_s S) = \mathbb{T}_{s,s}[-1]$

Ex] $\mathbb{T}_{s,s}$: $\mathbb{C}^2 \to \mathbb{C}$

$\mathbb{C}^2 \to \mathbb{C} \simeq \overset{0}{\mathbb{C}}$

$s \neq 0$

$\mathbb{C}^2 \to \mathbb{C}$

$s = 0$

$\mathbb{T}_{s,s}[-1] = \underset{X,Y}{\mathbb{C}^2[-1]} \oplus \underset{Z}{\mathbb{C}[-2]}$

$[X, Y] = Z$

$\Omega_s S \cong k \otimes_A k$

(is not k.!) in derived setting

2) Quasi-smooth schemes and scheme of singularities

$$\mathrm{Coh}\,X \overset{\text{different!}}{\longleftrightarrow} \mathrm{Perf}\,X$$

is from singular nature of X
not stacky nature

<u>Goal</u>: Find a reasonable class of singular schemes

<u>Prop1</u>: A derived scheme Z is smooth classical

$\Leftrightarrow$ $\mathbb{T}_Z$ is a vector bundle

$\Leftrightarrow$ $H^i(\mathbb{T}_{Z,z}) = 0 \quad \forall i > 0,\ z \in Z$

<u>Defn1</u>: A derived scheme is quasi-smooth if $\mathbb{T}_Z$ is perfect of amplitude $[0,1$

$(\Leftrightarrow H^i(\mathbb{T}_{Z,z}) = 0 \quad \forall i > 1,\ z \in Z)$

$$\mathbb{T}_Z|_U = \left(\mathcal{O}_Z^n|_U \to \mathcal{O}_Z^m|_U[-1]\right)$$

<u>Rmk1</u>: $\mathcal{X}$ a moduli space

$\Rightarrow$ want intersection theory of X

For that, we need $[\mathcal{X}]^{vir}$

In all the cases appearing in enumerative geometry,

$[\mathcal{X}]^{vir}$ arises from quasi-smoothness of $\mathcal{X}^{der}$ derived version of $\mathcal{X}$ "Perfect obstruction theory"

$\uparrow$

(It is believed)

ropl A derived scheme Z is quasi-smooth iff Z can be written (Zariski-locally)

as
$$\begin{tikzcd} Z \arrow[r] \arrow[d, "r"] & \mathbb{A}^n \arrow[d, "F"] \\ pt \arrow[r, "\{0\}"] & \mathbb{A}^m \end{tikzcd}$$

pf. $\Leftarrow$)
$$\begin{tikzcd} Z \arrow[r] \arrow[d] & U \arrow[d] \\ pt \arrow[r] & V \end{tikzcd}$$
U, V $\overset{\text{smooth}}{\wedge}$ classical schemes

$$\mathbb{T}_Z = \ker(df : \mathbb{T}_U|_Z \to \mathbb{T}_V|_Z)$$

$$\Rightarrow \mathbb{T}_Z = \ker(\mathcal{O}_Z^n \to \mathcal{O}_Z^m) \text{ Zariski-locally}$$

$$= \mathcal{O}_Z^n \longrightarrow \mathcal{O}_Z^m[-1]$$

$\square$

- $m=1$ ~> hypersurface

In particular, all hypersurfaces are quasi-smooth

- A qs derived scheme ⇔ locally complete intersection in a derived world
- A qs classical scheme ⇔ l.c.i. from regular sequence

$$\mathbb{T}_Z = (\mathbb{T}_U|_Z \xrightarrow{df} \mathbb{T}_V|_Z[-1])$$

$$\mathbb{L}_Z = (\mathbb{L}_V|_Z[1] \xrightarrow{df^*} \mathbb{L}_U|_Z)$$

If Z is smooth, df is surjective (df^* is inj) we only have $H^0(\mathbb{T}_Z)$, $H^0(\mathbb{L}_Z)$

If Z is not, we have $H^1(\mathbb{T}_Z)$, $H^{-1}(\mathbb{L}_Z)$

Z quasi smooth $\boxed{\text{classical}}$

$\rightsquigarrow$ $\mathrm{Sing}\, Z$ scheme of singularities

s.t. $\mathrm{Sing}\, Z$ measures how far Z is from being smooth

<u>Defn</u> $\mathrm{Sing}\, Z = \mathrm{Spec}_{Z^{cl}} \mathrm{Sym}_{H^0(\mathcal{O}_Z)} H^1(T_Z)$

$$\begin{tikzcd} \mathrm{Sing}\, Z \arrow[d] \\ Z^{cl} \end{tikzcd}$$

$= (T^*[-1] Z)^{cl}$

$\mathcal{O}_{T^*[n] Z} = \mathrm{Sym}_Z \, \Pi_Z[-n]$

for Z quasi-smooth

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{A}^n \arrow[d] \\ pt \arrow[r, "\{0\}"] & \mathbb{A}^m \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d] \\ pt \arrow[r, "\{0\}"] & V \end{tikzcd}$$

$\Pi_{V,pt} = V$

$\amalg_{V}/_Z \cong \mathcal{O}_Z \otimes V^*$

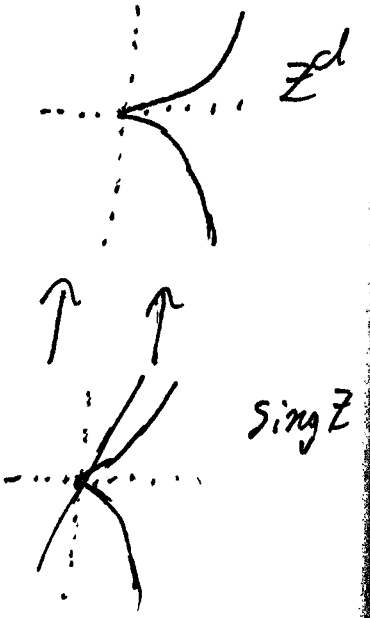
$\Rightarrow \mathrm{Sing}\, Z \subset Z^{cl} \times V^*$

$$\begin{tikzcd} \mathrm{Sing}\, Z \arrow[d] \\ Z^{cl} \end{tikzcd}$$

<u>Ex</u> $U \xrightarrow{f} \mathbb{A}^1$

$$\begin{tikzcd} Z = Z(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d, "f"] \\ \{0\} \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

$f = y^2 - x^3$



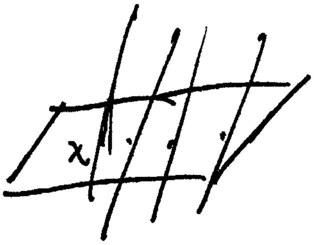
3) Singular support of coherent sheaves

$\underbrace{Coh(X)}$ vs. $Perf(X)$

A classical associative alg $A\text{-mod}^{\heartsuit}$

$U(\mathfrak{g})$-mod

A-mod



$Spec(Z\mathfrak{g})$

$Spec(ZA)$

Let $\mathcal{C}$ be a DG category.

Defn| The center of $\mathcal{C}$ is

$$\underset{\substack{\uparrow \\ \text{"Hochschild cochains"}}}{HC(\mathcal{C})} = End(\mathbb{1}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C})$$

$\{\varphi_c \in Hom(c,c)\}_{c \in \mathcal{C}}$ s.t. for $c \xrightarrow{f} c'$; $f \circ \varphi_c = \varphi_{c'} \circ f$

Ex| $\mathcal{C} = A$-mod

$\{\varphi_M\}_{M \in A\text{-mod}} \longleftrightarrow \varphi_A : A \to A \in End_A A$

central as we consider

$f_a : A \to A$

$\shortparallel$

$a\cdot$

$End_A(A)$

$$HH^\bullet(\mathcal{C}) = \bigoplus_{n\in\mathbb{Z}} H^n(HC(\mathcal{C}))$$

Hochschild coh of $\mathcal{C}$

$$HH^0(A\text{-mod}) \to Z(\mathrm{End}_A(A))$$

$\curvearrowleft$ $= Z(A^{op}) = Z(A)$

$\mathcal{C}$ DG category $\rightsquigarrow T = Ho(\mathcal{C})$

$\boxed{R \circlearrowleft \mathcal{C}}$ (e.g $\overset{H}{HC}(\mathcal{C})$)

$R \circlearrowleft T$

graded comm. algebra

$r \in R^{2n}$ $\quad r: t \to t[2n]$ $\quad \forall t \in T$

$\forall f: t \to t'$,

$$\begin{tikzcd} t \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & t' \arrow[d] \\ t[2n] \arrow[r, "f"] & t[2n] \end{tikzcd}$$

$R \to HH^\bullet(\mathcal{C}) \circlearrowleft T$

$\uparrow$

we want to find this!

<u>Thm</u> (Hochschild Konstant Rosenberg)

let X smooth affine scheme $/k$, char $k = 0$

$$HH^\bullet(QC(X)) = H^0(X, \Lambda^\bullet_{\mathcal{O}} T_X)$$

$\uparrow$ polyvector fields

note $QC(X) = \mathcal{O}_X\text{-mod}$
$= A\text{-mod}$
$HC(A) = Ext(A,A) = A \otimes A^{op}$

X quasi-smooth affine

$$HC(X) \simeq \Gamma(X, \mathcal{U}_{\mathcal{O}_X}(\mathbb{T}_X[-1]))$$

$\uparrow$ as associative algebra

$//$

$HC(QC(X))$

$\mathbb{T}_X[-1]$: Lie algebra in $QC(X)$

$\mathcal{U}$ universal enveloping algebra

If X is smooth, $\mathbb{T}_X = T_X$

$$\mathcal{U}_{\mathcal{O}_X}(\underset{\uparrow}{\mathbb{T}_X[-1]}) = \mathrm{Sym}_{\mathcal{O}_X} T_X[-1] =: \underset{\uparrow}{\Lambda^{\bullet}_{\mathcal{O}_X}} T_X$$

trivial lie alg $\mathcal{O}_X$

symmetric + deg 1 shifting

$\Gamma(X, \mathcal{O}_X) \to HC(X)$ ↰ module over

$\Gamma(X, \mathbb{T}_X[-1]) \to HC(X)$

$H^0(X, \mathcal{O}_X) \to HH^0(X)$ ↰ module

: quasi -smooth $H^1(X, \mathbb{T}_X) \to HH^2(X)$

/ $:= \mathrm{Sym}_{H^0(X,\mathcal{O}_X)}[H^{\bullet}(X, \mathbb{T}_X)] \to HH^{\bullet}(X) \to \mathrm{End}(\mathcal{F})$

ingZ

$\forall \mathcal{F} \in Coh(X)$

<u>Defn</u> $\mathcal{F} \in Coh(X)$

- $\mathrm{Sing\,Supp}\,\mathcal{F} = \mathrm{supp}_{\mathrm{Sing}\,X} \mathrm{End}\,\mathcal{F} \subset \mathrm{Sing}\,X$
- for $Y \subset \mathrm{Sing}\,X$, $Coh_Y(X) \subset Coh(X)$ is full subcategory consisting of sheaves $\mathcal{F}$ s.t. $\mathrm{Sing\,supp}\,\mathcal{F} \subset Y$.

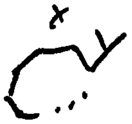
$$\begin{tikzcd}
 & QC(Flat_{G^\vee}) \arrow[d, phantom, "\cup" rotate=90] \arrow[dl, bend right, "\neq"'] & 0 \arrow[d, phantom, "\cup" rotate=90] \\
D(Bun_G) \arrow[r, "{[A-G]}", "\sim"'] \arrow[dr, hook, "\neq"'] & IC_N(Flat_{G^\vee}) \arrow[d, phantom, "\cup" rotate=90] & N/G \arrow[d, phantom, "\cup" rotate=90] \\
 & IC(Flat_{G^\vee}) & \mathfrak{g}^*/G
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & U \arrow[d] \\
pt. \arrow[r] & V
\end{tikzcd} \rightsquigarrow \quad Sing\ X \subset X \times V^*$$

$Loc_G \subset$ moduli of local systems

$\quad \| $

$\quad Hom(\pi_1(C), G)/G$



$xyx^{-1}y^{-1}$

$$\begin{tikzcd}
Hom(\pi_1(C), G) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & G^{2g} = U \arrow[d, "{[\,,\,]}"] \\
\{1\} \arrow[r] & G = V
\end{tikzcd}$$

$Sing\ Loc_G = Loc_G \times \mathfrak{g}^*/G$

$N_G = Loc_G \times N$

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathbb{A}^n \arrow[d] \\
\{pt\} \arrow[r] & \mathbb{A}^m
\end{tikzcd}$$

$n=0$

$W = \mathrm{Spec}\, k[\eta_1, \dots, \eta_m] \quad |\eta_i| = -1$

<u>Thm</u> (Koszul duality)

① $\mathrm{Ext}_{k[\eta]}(k,k) = k[\xi_1, \dots, \xi_m] \quad |\xi_i| = 2$

② $K : k[\eta]\text{-mod} \to k[\xi]\text{-mod}$

$\qquad M \longrightarrow \underline{\mathrm{Hom}}_{k[\eta]}(k, M)$

induces a fully faithful functor on $k[\eta]^{f.g.}\text{-mod}$

③ $\mathrm{coh}(W) \simeq k[\xi]\text{-mod}$

$$\begin{tikzcd}
\mathrm{Perf}(W) \arrow[r, hook] \arrow[d, squiggly] \arrow[d, phantom, "\partial_{coh_f(W)}", shift left=4ex] & QC(W) \arrow[d, squiggly] \\
\mathrm{coh}(W) \arrow[r, hook] & IC(W)
\end{tikzcd}
=
\begin{tikzcd}
k[\xi]\text{-mod}_0^{f.g.} \arrow[r, hook] \arrow[d, squiggly] & k[\xi]\text{-mod}_0 \arrow[d, squiggly] \\
k[\xi]\text{-mod}^{f.g.} \arrow[r, hook] & k[\xi]\text{-mod}
\end{tikzcd}$$

$\boxed{8}$ Revisiting $D(\mathrm{Bun}_G)$

conjecture

$$D(\mathrm{Bun}_G) \simeq IC_{\mathring{\mathcal{N}}_{\check{G}}}(\mathrm{Flat}_{\check{G}})$$

---

1) Conjecture for $G = \mathbb{G}_m$

It is known

$\mathrm{Bun}_{\mathbb{G}_m}$?

on $C = \mathbb{P}^1$
$\mathcal{L} = \mathcal{O}(n)$
$n \in \mathbb{Z}$

Recall $\underline{\mathrm{Pic}}(C)$ Picard scheme

$\underline{\mathrm{Pic}}^0(C)$ deg 0 part

$\cong \mathrm{Jac}\, C$

want to write
$\mathrm{Bun}_{\mathbb{G}_m} \simeq \mathrm{Pic}\, C$

$$\underset{\substack{\uparrow \\ \text{non-canonical}}}{\simeq} \mathrm{Jac}\, C \times \mathbb{Z} \times B\mathbb{G}_m$$

$G = \mathbb{G}_m \longleftrightarrow \mathbb{G}_m = \check{G}$

$G = GL_n \longleftrightarrow GL_n = \check{G}$

$$\mathrm{Flat}_{\mathbb{G}_m} \underset{\substack{\uparrow \\ \text{non-canonical}}}{\simeq} \underline{\mathrm{Flat}}_1 \times B\mathbb{G}_m \times \mathrm{Spec}\, k[\eta]$$

$\deg \eta = -1$

$\mathbb{T}\, \mathrm{Flat}_{\mathbb{G}_m} \simeq (\Omega^{\bullet}, d_{dR})$ in smooth cat.

$$= \left(\overset{-1}{\Omega^0} \to \overset{0}{\Omega^1} \to \overset{1}{\Omega^2} \cdots\right)$$

$\mathcal{N} \subset \mathfrak{g}^*/G$

$G = T$

$\Rightarrow \mathcal{N} = 0$

$\Rightarrow IC_{\mathcal{N}} = QC$

<u>Claim</u>

$$D(Jac) \simeq QC(Flat_1)$$

$$D(\mathbb{Z}) \simeq QC(\mathbb{B}G_m)$$

$$D(\mathbb{B}G_m) \simeq QC(Spec\ k[\eta])$$

① $T^* Jac \simeq Jac \times H^0(C, \Omega_C)$

$$QC(T^* Jac) \underset{FM}{\cong} QC(T^* Jac)$$

$$\Big\downarrow \text{deformation}$$

$$D(Jac) \underset{FMLR}{\cong} QC(Flat_1)$$

② ~~$\mathbb{B}G$~~ $D(\mathbb{Z}) = \mathbb{Z}$-graded vect

Claim $QC(\mathbb{B}G) \simeq Rep\ G$

$\rightsquigarrow QC(\mathbb{B}G_m) \simeq Rep\ G_m$

③ <u>rhs</u> $= k[\eta]$-mod

<u>Goal</u> understand $D(\mathbb{B}G_m)$

Recall $\mathcal{X}$ prestack

$$D^{l}(\mathcal{X}) = QC(\mathcal{X}_{dR})$$

where $\mathcal{X}_{dR}(S) := \mathcal{X}(S^{red})$

category of left D-modules

Defn $D^{r}(\mathcal{X}) := IC(\mathcal{X}_{dR})$

// right D-modules

$$D^{l}(\mathcal{X}) \xrightarrow{r_{\mathcal{X}}} D^{r}(\mathcal{X}) \qquad \text{where } \omega_{\mathcal{X}} = p_{\mathcal{X}}^{!} k$$

$$\mathcal{F} \longmapsto \mathcal{F} \otimes \omega_{\mathcal{X}} \qquad \text{for } p_{\mathcal{X}} : \mathcal{X} \to pt$$

Six functors formalism

$f : X \to Y$

(f^*, f_*) $(f_!, f^!)$ adjoint pairs $\otimes$, Hom

pullback

$\int$ along fiber

$\int$ of cpt. supp of fiber

Verdier duality $\mathbb{D}$

$f^* = \mathbb{D} f^! \mathbb{D}$

- If f is proper then $f_! = f_*$
- If f is smooth of relative real dim d $f^! = f^*[d]$

Review of Koszul duality

$$\begin{tikzcd} Z_\Gamma \arrow[r] \arrow[d] & \mathbb{A}^n \arrow[d] & & \operatorname{spec} k[\eta] \arrow[r] \arrow[d] & pt \arrow[d] \\ pt \arrow[r, "\{0\}"] & \mathbb{A}^m \arrow[r, phantom, "\rightsquigarrow"] & & pt \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

$|\eta| = -1$

$\left[k \otimes^L_{k[x]} k = k[\eta] \right]$

Prop) $QC(\operatorname{Spec} k[\eta]) = k[\eta]\text{-mod} \xrightarrow{KD} \overset{\text{Vect}}{k[\xi]}\text{-mod}$

$$M \longrightarrow \underline{\operatorname{Hom}}_{k[\eta]}(k, M)$$

① $k \longrightarrow \underline{\operatorname{Hom}}_{k[\eta]}^{\circlearrowleft}(k,k) = k[\xi]$

conclude $KD : k[\eta]\text{-mod} \to k[\xi]\text{-mod}$

~~Hom~~ $V = k\langle x \rangle$ ($\uparrow$ $x_1, \dots x_m$) $\quad W = V[1]$

$$\underline{\operatorname{Hom}}_{\operatorname{Sym} W}(k,k)$$

$$k \simeq \operatorname{Sym}(W[1] \oplus W) \qquad (d = id : W[1] \to W)$$

$\overset{-1}{\mathbb{C}^2} \xrightarrow{d=id} \overset{0}{\mathbb{C}^2}$ $\Rightarrow$ trivial

$[\dots \to \operatorname{Sym} W \otimes \operatorname{Sym}^2 W[1] \to \operatorname{Sym} W \otimes W[1] \to \operatorname{Sym} W \to k]$ Koszul resolution

$$\underline{\operatorname{Hom}}_{\operatorname{Sym} W}(\operatorname{Sym}(W[1] \oplus W), k)$$
$$= \underline{\operatorname{Hom}}(\operatorname{Sym}(W[1]), k) = \operatorname{Sym}(W^*[-1])$$
$$= k(\xi)$$

② KD is fully faithful on k

pf: $\operatorname{Hom}_{k[\eta]}(k,k) = k[\xi] = \operatorname{Hom}_{k[\xi]}(k[\xi], k[\xi])$

Conclude KD is fully faithful on $\operatorname{Perf}(\operatorname{Spec} k[\eta])$

$\mathrm{Perf}(\mathrm{Spec}\ k[\xi])$
is f.g. over k
because $k[\eta]$ is Artinian

$$\begin{tikzcd}
\text{f.g. over } k \arrow[r, "\sim"] & \text{f.g. over } k[\xi] \\
\mathrm{Perf}(\mathrm{Spec}\ k[\eta]) \arrow[u, phantom, "\cup"] \arrow[ur, hook] &
\end{tikzcd}$$

---

③ $\mathrm{Coh}(\mathrm{Spec}\ k[\eta]) \xrightarrow{\sim} k[\xi]\text{-mod}^{f.g.}$

Summary To understand $QC(\mathrm{Spec}\ k[\eta])$
we find a generator k and show
$QC(\quad) \sim \mathrm{Hom}(k,k)\text{-mod}$

Rmk (noncommutative geometry)

Fuk | Coh
X symp | X CY-manifold

$$\begin{tikzcd}
\text{Try} & \mathrm{Coh}(X) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\|"] & A\text{-mod} \\
& \mathcal{O}_X\text{-mod} & \mathrm{Ext}(g_1 \oplus \cdots \oplus g_n) \arrow[u] \\
& & \text{self-ext} \arrow[u]
\end{tikzcd}$$

To understand $D(BG_m)$ what should we do?
need to find a nontrivial obj
claim! local systems on BG_m are all trivial
$\pi_1(BG) = \pi_0(G)$ $\Rightarrow$ $\pi_1(BG_m) \xrightarrow[\text{trivial}]{} GL_n$
$\pi_0(G_m) = \{1\}$

$$\cdots \mathbb{G}\times\mathbb{G}\times\mathbb{G} \;\substack{\to\\\to\\\to\\\to}\; \mathbb{G}\times\mathbb{G} \;\substack{\to\\\to\\\to}\; \mathbb{G} \rightrightarrows B\mathbb{G}$$

$\omega_{B\mathbb{G}_m} = p^! k \qquad p : B\mathbb{G}_m \to pt$

$k \leftrightarrow \omega$

$$\mathrm{Hom}_{D(B\mathbb{G}_m)}(\omega_{B\mathbb{G}_m}, \omega_{B\mathbb{G}_m}) = dR(B\mathbb{G}_m)$$
$$\simeq k[\xi]$$
$$|\xi| = 2$$

$D(B\mathbb{G}_m) \to k[\xi]\text{-mod}$

$\omega \longrightarrow \mathrm{Hom}(\omega, \omega)$

$M \mapsto \underline{\mathrm{Hom}}_{k[\eta]}(k, M)$

equiv over M f.g.

$0 \to \omega[1] \to g \to \omega \to$

$QC(\mathrm{Spec}\ k[\eta])$

cpt. generator $k[\eta] \underset{\text{vector space}}{\simeq} k \oplus k[1]$

$\Big\Vert$ - - - - $\updownarrow$ $\uparrow$ $\mathrm{Ext}^1(k, k[1]) = \mathrm{Ext}^2(k, k)$

$\nearrow$ $D(B\mathbb{G}_m)$ — $g \simeq \omega \oplus \omega[1]$ — —

$\mathrm{Hom}(g, -)$ cpt. generator

$\xi \in \mathrm{Ext}^2(\omega, \omega)$
$= \mathrm{Ext}(\omega, \omega[1])$

$$\mathrm{Hom}_{D(B\mathbb{G}_m)}(g, g)\text{-mod} \simeq QC(\mathrm{Spec}\ k[\eta])$$
$$\Vert$$
$$k[\eta]$$

$QC(\operatorname{Spec} k[\eta])$	$D(B\mathbb{G}_m)$
k	ω
$k[\eta]$	g
$\operatorname{Hom}_{k[\eta]}(k[\eta], k[\eta]) = k[\eta]$	$\operatorname{Hom}_{D(B\mathbb{G}_m)}(g, g) = k[\eta]$
$\operatorname{Hom}_{k[\eta]}(k, k) = k[\xi]$	$\operatorname{Hom}_{D(B\mathbb{G}_m)}(\omega, \omega) = k[\xi]$

$\mathrm{Bun}_G = G(k(C)) \backslash G(\mathbb{A}) / G(\mathcal{O})$

G semisimple

G-bundle on C

can be understood as triv. G-bundles P_1 on D_x

+ triv G-bundle P_2 on $C \setminus \{x\}$ and identification

$$P_1|_{D_x^\times} = P_2|_{D_x^\times}$$

$\mathrm{Gr}_G = G(K)/G(\mathcal{O})$ affine Grassmannian

$K = \mathbb{C}((t))$

$\mathcal{O} = \mathbb{C}[[t]]$

moduli space of G-bdls P on D w/ trivializations $P|_{D^\times} \cong P^0|_{D^\times}$

$G = GL_n$

$Gr_G \simeq \{\text{lattices in } K^n\}$

$L \subset K^n$

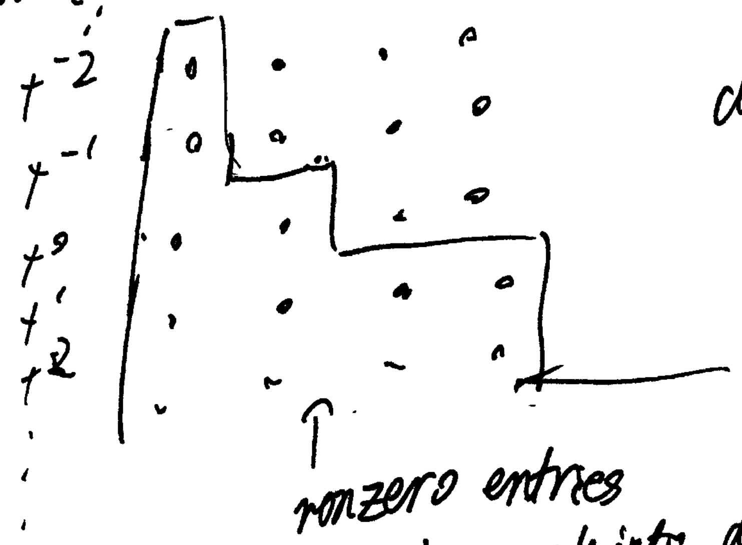
$t \cdot L \subset L$

pf) $\mathcal{O}^n \subset K^n$ lattice

$G(K)$ transitive action $\quad t^N\mathcal{O}^N \subset L \subset t^{-N}\mathcal{O}$

$G(\mathcal{O})$ stabilizer $\quad \forall N \gg 0$

Picture $n=4$.



dots = basis of lattice

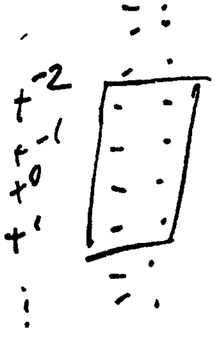
lattice

nonzero entries (can be made into descending stairs post. permutation)

$Gr_{GL_n} = \bigcup_{k \in \mathbb{Z}_{\geq 0}} Gr_k$

where Gr_k one can take $N=k$ ~~in the tightest way~~

$n=2 \quad k=2$



$t^{-2}\mathcal{O}^2 / t^2\mathcal{O}^2$

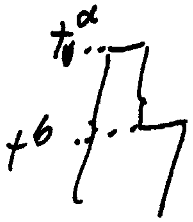
$= \mathbb{C}^8$

$t = \left(\begin{array}{c|c} \begin{smallmatrix}0&1&&\\&0&1&\\&&0&\ddots\end{smallmatrix} & 0 \\ \hline 0 & \begin{smallmatrix}0&1&&\\&0&1&\\&&0&\ddots\end{smallmatrix}\end{array}\right)$

Gr_G ind-proj

(colim of proj. var)

$Gr_k = \coprod_{-k\le a,b\le k} Gr_{a,b}$



$t^{b}L \subset t^a\mathcal{O}^2$

$t^{b-1}\mathcal{O}^2 \not\subset L \not\subset t^{a+1}\mathcal{O}^2$

<u>Prop</u> $\overline{Gr_{a,b}} = \coprod_{0\le i\le \frac{1}{2}(b-a)} Gr_{a+i,b-i}$

pf) $\alpha_{a,b} = \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix} \in Gl_2(K)$

find a sequence

s.t. $\{\alpha_k\}_{k=1}^{\infty} \to \alpha_{a+i,b-i}$

$\uparrow$

$Gr_{a,b}$

$$\alpha_k = \langle \underset{t^a\mathcal{O}^2}{\underset{\cap}{t^{a+i}e_1}}, \overset{t^b\mathcal{O}^2}{\overset{\cap}{\tfrac{1}{k}t^a + t^{b-i}e_2}} \rangle$$

consider $\Lambda^2 : Gr_{Gl_2} \to Gr_{Gl_1} = \mathbb{Z}$

$\alpha_{a,b} \to t^{a+b}\mathcal{O}$

$\Rightarrow \coprod_{a+b=j} Gr_{a,b}$ is closed ☒

Gr_{GL_2} connected components?

$\coprod_{i\geq 0} Gr_{a-i,a+i}$ $\quad \coprod_{i\geq 0} Gr_{a-i,a+1+i}$ $\quad a\in\mathbb{Z}$

$\pi_0(Gr_{GL_2}) = \mathbb{Z}$

Gr_{SL_2} cpt. $= \coprod_{i\geq 0} Gr_{-i,i}$

$\pi_0(Gr_{SL_2}) = \{1\}$ $\quad \pi_0 = \mathbb{Z}_2$

$Gr_{PGL_2} = \coprod_{i\geq 0} Gr_{0,2i}\ ,\ \coprod_{i\geq 0} Gr_{0,2i+1}$

$$\pi_0(Gr_G) = \pi_1(G) \underset{\underset{G \text{ simple}}{\uparrow}}{=} Z(\check{G})$$

$G = SL_2 \Rightarrow 1 = \pi_1(SL_2) \cong Z(PGL_2)$

$G = PGL_2 \quad \pi_1(PGL_2) = C_2 \cong Z(SL_2)$

$Gr_G \simeq \Omega K$

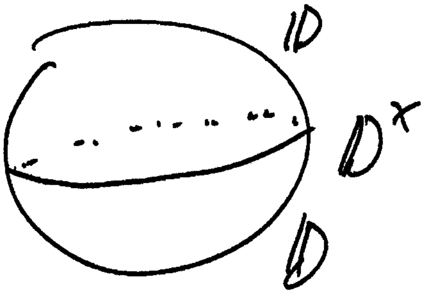
$K \underset{\text{cpt gp.}}{\subset} G$

$GL_2 \rightsquigarrow Gr_{a,b}$

G reductive gp $\rightsquigarrow \lambda \in \Lambda_G^+$ dominant coweights

$Gr_G = \coprod_{\lambda\in\Lambda_G^+} Gr_\lambda$ schubert varieties

$\nwarrow G(\mathcal{O})$ - equivariant orbit



$$\mathrm{Bun}_G(\mathbb{P}^1) \simeq G(k[t^{-1}]) \backslash G(k((t))) / G(k[[t]])$$

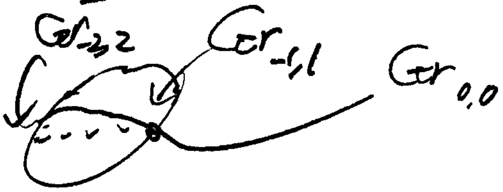
$$G = GL_n$$

Thm (Birkhoff, Grothendieck) any vector bundle of rank n over $\mathbb{P}^1$ is iso to $\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_n)$

connected components $\wedge$ $\sum k_i$
is parameterized by

$\mathcal{O} \oplus \cdots \oplus \mathcal{O} \in$ cmpt. w/ $\sum k_i = 0$

Gr_{GL_2}



$\cdots$ $\mathcal{O}(-2) \oplus \mathcal{O}(2)$ $\quad$ $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ $\quad$ $\mathcal{O} \oplus \mathcal{O}$

in $\mathrm{Bun}_{GL_2} \mathbb{P}^1$

$\mathcal{O} \oplus \mathcal{O}$ is open dense

9 How to study $D(\mathrm{Bun}_G)$

<u>Last time</u> $G = \mathbb{G}_m = GL_1$

$$\begin{array}{ccccccc} D(\mathrm{Bun}_{\mathbb{G}_m}) & \simeq & D(\mathrm{Jac}) \otimes & D(\mathbb{Z}) & \otimes & D(B\mathbb{G}_m) \\ \wr\wr & & \wr\wr & \wr\wr & & \wr\wr \\ QC(\mathrm{Flat}_{\mathbb{G}_m}) & \simeq & QC(\underline{\mathrm{Flat}_1}) \otimes & QC(\mathbb{G}_m) & \otimes & QC \end{array}$$

For $G = T$ torus

$$\mathrm{Bun}_T = \coprod_{X} \mathrm{Jac}_T \times B\mathbb{G}_m$$

$X \leftarrow$ cocharacter lattice

$= \{ \cancel{V}\, \mathbb{G}_m \to T \mid \text{homo} \}$

<u>Fact</u> $\pi_0(\mathrm{Bun}_G) \simeq \pi_0(\mathrm{Gr}_G) = \pi_1(G)$

$$= \underset{\substack{\uparrow \\ \text{coweights}}}{\Lambda_G} \Big/ \underset{\substack{\uparrow \\ \text{coroots}}}{\check{R}_G}$$

$\Rightarrow \pi_0(\mathrm{Bun}\, T) = \pi_1(T) = \Lambda_T$

How about Bun_G?

G is combinatorially complicated object and there is no such easy description

Idea: use easier groups attached to G

$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \leftrightarrow B$ Borel subgroup (solvable)

$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \leftrightarrow N$ unipotent radical

1) Generic Reductions

$$\mathrm{Bun}_G(C) : (\mathrm{Sch}^{\mathrm{aff}})^{op} \to \mathrm{Spc}$$

$$S \to \{\underbrace{P_G \text{ on } C \times S = \mathcal{G}}_{\text{Principle } G\text{-bundle}}\}$$

scheme $X \rightsquigarrow \mathrm{Hom}(-, X)$

$$\mathrm{Bun}_G C \rightsquigarrow \mathrm{Hom}(-, \mathrm{Bun}_G C)$$
$$= \mathrm{Hom}(-, \mathrm{Hom}(C, BG))$$
$$= \mathrm{Hom}(C \times (-), BG)$$

$\uparrow$

definition of Bun_G in Alg. Geom note $BG: (Sch^{aff})^{op} \to Spc$

$S \to Bun_G(S)_{spc}$ ← no idea about algebraic structure

$C \to Bun_G(C)_{spc}$ ←

definition of Bun_G in Alg. Top

$Bun_B : S \mapsto \{P_B \text{ on } C_S\}$

Plücker description of Bun_B

B-bundle + flag section is flag of bundles

$B \hookrightarrow G$ ~~BB~~

$$\begin{tikzcd}
G/B \arrow[r] \arrow[d, "\Gamma"'] & \mathbb{B}B = pt/B \arrow[d] \\
pt \arrow[r] & \mathbb{B}G = pt/G \\
C \arrow[ur, bend left, to=1-2] &
\end{tikzcd}$$

$G = Gl_n$ $V=$ defining rep

$W \subset V$ dim k

$\Lambda^k W \subset \Lambda^k V$ or $\Lambda^k W \in \mathbb{P}(\Lambda^k V)$

line

$Gr(k,n) \xhookrightarrow{} \mathbb{P}(\Lambda^k V)$

↑ Plücker embedding

$W_1 \subset \cdots \subset W_n = V$

$$Fl(V) \hookrightarrow \prod_{k=1}^{n} \mathbb{P}(\Lambda^k(V))$$

$$\begin{tikzcd}
\Lambda^i V \otimes \Lambda^0 V \arrow[rr, "f_{ij}"] \arrow[rr, phantom, "\circlearrowright"', shift right=3] \arrow[dr] & & \Lambda^{i-1} V \otimes \Lambda^{j+1} V \\
& \Lambda^{i-1} V \otimes V \otimes \Lambda^j V \arrow[ur] &
\end{tikzcd}$$

$f_{11}(v_1 \otimes v_2) = v_1 \wedge v_2$

$f_{12}(v \otimes (v_1 \wedge v_2)) = v \wedge v_1 \wedge v_2$

$f_{22}((v_1 \wedge v_2) \otimes (v_3 \wedge v_4)) = v_1 \otimes (v_2 \wedge v_3 \wedge v_4) - v_2 \otimes (v_1 \wedge v_3 \wedge v_4)$

<u>Claim</u>/ $\{L_i \subset \Lambda^i V\}$ come from $Fl(V)$ lines

$\iff f_{ij}(L_i \otimes L_j) = 0$

$f_{11}(L_1 \otimes L_1) = L_1 \wedge L_1 = 0$

$\dim V = 4 \qquad v = \sum_{i<j} a_{ij} v_i \wedge v_j \in \Lambda^2 V$

$f_{22}(v, v) = a_{12} a_{34} - a_{13} a_{24} + a_{14} a_{23} = 0$

$Gr(2,4)$

$E_1 \subset \cdots \subset E_n = E \qquad G = GL_n$

$$\mathrm{Bun}_B = \{E_1 \subset \cdots \subset E_n = E\}$$

$$= \{\underset{\text{sub-bundles}}{L_i \subset \Lambda^i E} \mid f_{ij}(L_i \otimes L_j) = 0\}$$

$$G/B \hookrightarrow \prod_{\substack{\omega_i \\ \text{fundamental} \\ \text{weights}}} \mathbb{P}(V^{\omega_i})$$

$$\begin{tikzcd} & & BT \arrow[d, "\lambda"] \\ C \arrow[urr, "P_T"] \arrow[rr, "\mathcal{L}^{\lambda}_{P_T}"'] \arrow[rr, phantom, shift left=3, "\circlearrowleft"] & & BG_m \end{tikzcd}$$

$\lambda : T \to G_m$

$\mathrm{Bun}_B(S)$

$\Leftrightarrow \left\{ \begin{array}{l} P_G \text{ on } C_S \\ P_T \text{ on } C_S \\ \kappa^{\lambda} : \mathcal{L}^{\lambda}_{P_T} \hookrightarrow V^{\lambda}_{P_G} \quad \text{for } \lambda \in \Lambda^{\lambda}_G \text{ dominant} \end{array} \right.$

$V^{\lambda}_{P_G} \cong P_G \times_G V^{\lambda}$

s.t.

$$\begin{tikzcd} \mathcal{L}^{\lambda+\mu}_{P_T} \arrow[r, phantom, "\simeq"] \arrow[d] & \mathcal{L}^{\lambda}_{P_T} \otimes \mathcal{L}^{\mu}_{P_T} \arrow[r] & V^{\lambda}_{P_G} \otimes V^{\mu}_{P_G} \\ V^{\lambda+\mu}_{P_G} \arrow[urr, dash] & & \end{tikzcd}$$

$$\mathrm{Bun}_B \xrightarrow{p} \mathrm{Bun}_G$$

$$(P_G, P_T, \chi) \longmapsto P_G$$

$$P_B \longmapsto P_B \times_B G$$

Question/ $p^!: D(\mathrm{Bun}_G) \to D(\mathrm{Bun}_B)$ is Fully Faithful?

Answer No

$$\begin{tikzcd}
\mathrm{Bun}_B \arrow[r] \arrow[dr, bend right, "p_{\mathrm{Bun}_B}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Bun}_G \arrow[d, "p_{\mathrm{Bun}_G}"] \\
& pt
\end{tikzcd}$$

$\omega_{\mathcal{X}} = p_{\mathcal{X}}^! k \qquad p_{\mathcal{X}} : \mathcal{X} \to pt$

$$\omega_{\mathrm{Bun}_G} \to \omega_{\mathrm{Bun}_B}$$

$$\underline{\mathrm{Hom}}(\omega_{\mathrm{Bun}_G}, \omega_{\mathrm{Bun}_G}) \overset{?}{\simeq} \mathrm{Hom}(\omega_{\mathrm{Bun}_B}, \omega_{\mathrm{Bun}_B})$$

Take H^0: $H^0_{dR}(\mathrm{Bun}_G) \overset{?}{\simeq} H^0_{dR}(\mathrm{Bun}_B)$

$$B = T\underset{\substack{\sim \\ \text{contractible}}}{N} \Rightarrow H^0(\mathrm{Bun}_B) = H^0(\mathrm{Bun}_T)$$

$$\left.\begin{array}{l} \pi_0(\mathrm{Bun}_G) = \pi_1(G) \\ \pi_0(\mathrm{Bun}_B) = \pi_1(T) \end{array}\right\} \text{NOT SAME!}$$

(e.g. for GL_2 $\mathbb{Z} \neq \mathbb{Z}^2$)

$B \hookrightarrow G$

$B \to T = B/N$

$$\mathrm{Bun}_B \xrightarrow{q} \mathrm{Bun}_T$$

$$P_B \longmapsto P_B \times_B B/N$$

<u>Goal:</u> Enhance p to find a fully-faithful embedding

<u>Idea:</u> (From number theory)

for $k = \mathbb{F}_q$

$$\mathrm{Bun}_G(k) \simeq G(k(C)) \backslash G(\mathbb{A}) / G(\mathbb{O})$$

In top'l context

$$\mathrm{Bun}_G = L_{out}G \backslash LG / L_+G$$

where $LG =$  $\to G$

$L_+G =$  $\to G$

$L_{out}G =$  $\to G$

$$G(\mathbb{A})/G(\mathbb{O})$$

$$\mathbb{A} = \prod^{res}_{x\in C} K_x \qquad K_x = k((t))$$

$$\mathbb{O} = \prod_{x\in C} \mathcal{O}_x \qquad \mathcal{O}_x = k[[t]]$$

$$G(K_x)/G(\mathcal{O}_x) = Gr_G$$

$$\begin{tikzcd}
 & \boxed{?} \arrow[dl] & \\
Bun_G \arrow[d, equal] & \circlearrowleft & Bun_B \arrow[ul] \arrow[ll] \arrow[d, equal] \\
G(k)\backslash G(\mathbb{A})/G(\mathbb{O}) & \circlearrowright & B(k)\backslash B(\mathbb{A})/B(\mathbb{O}) \arrow[ll] \arrow[dl] \\
 & B(k)\backslash G(\mathbb{A})/G(\mathbb{O}) \arrow[ul] &
\end{tikzcd}$$

Iwasawa decomposition

$$G(K_x) = B(K_x)\, G(\underset{\substack{\uparrow \\ k[[t]]}}{\mathcal{O}_x})$$

$$k((t)) \to \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$\rightsquigarrow G(\mathbb{A})/G(\mathbb{O}) = B(\mathbb{A})/B(\mathbb{O})$$

$H \subset G$ subgroup

$\mathrm{Bun}_G^{H\text{-gen}}$ is a prestack

$$S \longmapsto (\mathcal{P}_G, U, \alpha_H)$$

Where $\mathcal{P}_G$ is a principal G-bundle on C_S

$U \subset S_S$ is an open set

α_H is a reduction

$$\begin{tikzcd}
 & BH \arrow[d] & \\
U \arrow[ur, dashed, "\alpha_H"] \arrow[r] & BG & \rightsquigarrow \mathcal{P}_{H,U}
\end{tikzcd}$$

$(\mathcal{P}_G, U, \alpha_H) \sim (\mathcal{P}_G', U', \alpha_H')$

if on $U \cap U'$ they are isomorphic

$$\begin{tikzcd}
 & (\mathcal{P}_G, C, \alpha_H) & \\
\mathcal{P}_H \arrow[ur, mapsto] & \mathrm{Bun}_G^{H\text{-gen}} \arrow[dr, bend left, "\mathfrak{p}_H^{enh}"] \arrow[d, phantom, "\circlearrowleft"] & \\
\mathrm{Bun}_H \arrow[ur, "i_H"] \arrow[rr, mapsto] & & \mathrm{Bun}_G \quad \mathcal{P}_G \\
\mathcal{P}_H & &
\end{tikzcd}$$

<u>Prop 1</u> $i_H : \mathrm{Bun}_H \to \mathrm{Bun}_G^{H\text{-gen}}$

induces an equivalence at the level of

k-pts, if H is parabolic

A parabolic subgroup P is anything in between $B \subseteq P \subseteq G$

<u>Pf</u> $U \to G/H$

can this extend to C?

G/H ~~is~~ proper if H is parabolic

valuative criterion of properness ☒

$$p_B^{enh} : \mathrm{Bun}_G^{B\text{-gen}} \to \mathrm{Bun}_G$$

$$\mathrm{Bun}_G^{N\text{-gen}}$$

$$\underline{Map}(C,T)^{gen}$$

$$S \longmapsto \left\{ \begin{array}{ccc} U & \xrightarrow{\gamma} & T \\ \cap & & \\ S_S & & \end{array} \right.$$

generic maps

$$\begin{tikzcd} T \arrow[r] \arrow[d] & BN \arrow[d] \\ pt \arrow[r] & BB \end{tikzcd} \Rightarrow \mathrm{Bun}_G^{N\text{-gen}} / \underline{Map}(C,T)^{gen} = \mathrm{Bun}_G^{B\text{-gen}}$$

<u>Thm</u> (J. Barlev) $(p_B^{enh})^! : D(\mathrm{Bun}_G) \longrightarrow D(\mathrm{Bun}_G^{B\text{-gen}})$

is fully faithful

Pf)

$$\begin{tikzcd}
\mathrm{Bun}_G^{1\text{-gen}} \times \mathrm{Map}(C, G/B)^{gen} \arrow[r] \arrow[d] & \mathrm{Bun}_G^{B\text{-gen}} \arrow[d, "\mathrm{Pen}^H"] \\
\mathrm{Bun}_G^{1\text{-gen}} \arrow[r] & \mathrm{Bun}_G
\end{tikzcd}$$

$$\begin{tikzcd}
G(\mathbb{A})/G(\mathbb{O}) \times G(K)/B(K) \arrow[r] \arrow[d] & B(K)\backslash G(\mathbb{A})/G(\mathbb{O}) \arrow[d] \\
G(\mathbb{A})/G(\mathbb{O}) \arrow[r] & G(K)\backslash G(\mathbb{A})/G(\mathbb{O})
\end{tikzcd}$$

<u>Claim</u>| $\mathrm{Map}(C, G/B)^{gen}$ is trivial in some sense

<u>Thm</u>| (Gaitsgory) If Y is nice (e.g. G/B, T)

$\Rightarrow$ $\mathrm{Map}(C, Y)^{gen}$ is homologically contractible

$\Rightarrow$

So now:

$$\begin{tikzcd}
 & \mathrm{Bun}_G^{B\text{-gen}} \arrow[dr] & \\
\mathrm{Bun}_B \arrow[ur] \arrow[rr] & & \mathrm{Bun}_G
\end{tikzcd}$$

for $G = GL_2$

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[dr] & \\
\pi_0 : \mathbb{Z}\times\mathbb{Z} \arrow[ur, "glued"] \arrow[rr] & & \mathbb{Z}
\end{tikzcd}$$

how does this gluing happen?

Ex $G = GL_2$

$$\mathrm{Bun}_B = \left\{ \underset{\text{sub-bundle}}{L \hookrightarrow E} \right\}$$

E/L bundle

$C = \mathbb{P}^1$

$$L = \mathcal{O} \underset{(x, x+t)}{\xrightarrow{\varphi_t}} E = \mathcal{O}(1) \oplus \mathcal{O}(1)$$

family of maps under t

For $t \neq 0$

$$0 \to \mathcal{O} \xrightarrow{h \mapsto (xh,\ (x+t)h)} \mathcal{O}(1) \oplus \mathcal{O}(1) \xrightarrow{f,g \mapsto (x+t)f - xg} \mathcal{O}(2) \to 0$$

for $t = 0$

$$0 \to \mathcal{O} \to \mathcal{O}(1) \oplus \mathcal{O}(1) \to \mathcal{O}(1) \oplus k_0 \to 0$$

in affine coordinates

$$0 \to k[x] \xrightarrow{h \mapsto (xh,\ xh)} k[x] \oplus k[x] \xrightarrow{(f,g) \mapsto (f-g,\ g(0))} k[x] \oplus \overset{\overset{k[x]/(x)}{\|}}{k} \to 0$$

$$\pi_0(\mathrm{Bun}_B) = \pi_0(\mathrm{Bun}_T) = \underset{\deg L}{\mathbb{Z}} \times \underset{\deg E/L}{\mathbb{Z}}$$

$t \neq 0 \Rightarrow (0, 2)$

$t = 0$ if one ignores $x = 0$, $(1, 1)$ "same component"

differ by $(1, -1)$ coroot

$$\mathrm{Bun}_B \xrightarrow{p} \mathrm{Bun}_G$$

p is not proper!

Drinfeld compactification

$\rightarrow$ $\overline{\mathrm{Bun}}_B$ compactification

$(p_G, p_T, \kappa^\lambda)$

embedding of coh sheaves

then $\overline{\mathrm{Bun}}_B \xrightarrow{\bar{p}} \mathrm{Bun}_G$ is proper

---

2) Global overview

$$\underbrace{\boxed{D(\mathrm{Bun}_G) \overset{?}{\simeq} IC_{N_{\check{G}}}(\mathrm{Flat}_{\check{G}})}}_{\text{The conjectured Geometric Langlands Correspondence}}$$

ext week

hardest part of GLC!

$$\begin{tikzcd} \boxed{\mathrm{Whit}^{ext.}(G)} \arrow[r, leftrightarrow] & \mathrm{Glue}(\check{G}) \\ D(\mathrm{Bun}_G) \arrow[u, hook, "!"] \arrow[r, phantom, "\overset{?}{\simeq}"] & IC_{N_{\check{G}}}(\mathrm{Flat}_{\check{G}}) \arrow[u, hook, "{[AG2]}"'] \end{tikzcd}$$

[AG1]

POINT The upper two categories are of $\underbrace{\text{local}}_{\substack{\text{to be explained}\\ \text{in the final lecture:}\\ \text{on Geometric Satake}}}$ nature!

$$D(\mathrm{Bun}_G) \overset{?}{\simeq} IC(\mathrm{Flat}_{\check{G}})$$

KM-rep / oc(OP$_{\check{G}}$) opers

Conformal Field theory (via. Belinson-Drinfeld - Hitchin System)

Number theory: $M \leftarrow P \rightarrow G$

$$\begin{tikzcd} & \mathrm{Bun}_P \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Bun}_M & & \mathrm{Bun}_G \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Flat}_{\check{G}} \arrow[dl, "q^{Gal}"'] \arrow[dr, "p^{Gal}"] & \\ \mathrm{Flat}_{\check{M}} & & \mathrm{Flat}_{\check{G}} \end{tikzcd}$$

⇒ Can define $\overset{\substack{\text{geometric Eisenstein functor}\\ \downarrow}}{\mathrm{Eis}_M(\mathcal{F})} = p_! q^* \mathcal{F}$ on D! | $\mathrm{Eis}_M^{Gal}(\mathcal{F}) = p_*^{Gal}(q^{Gal})^! \mathcal{F}$ on IC

$$\begin{tikzcd}
D(Bun_G) \arrow[r, phantom, "\underset{\mathbb{L}_G}{\overset{?}{\simeq}}"] & IC_{N_{\check{G}}}(Flat_{\check{G}}) \\
D(Bun_M) \arrow[u, "Eis_M"] \arrow[r, phantom, "\underset{\mathbb{L}_M}{\simeq}"] & IC_{N_{\check{M}}}(Flat_{\check{M}}) \arrow[u, "Eis_M^{Gal}"']
\end{tikzcd}$$

Kac-Moody rep.s + Eis generate D	opers + Eis^{Gal} generate $IC_{N_{\check{G}}}$

Everything here has interpretation under

$\mathcal{N} = 4$ SUSY Yang-Mills

$\boxed{10}$ Factorization Structures

Goal: Understan $D(\mathrm{Bun}_G)$

Recall $$D(\mathrm{Bun}_G) \underset{\substack{\uparrow \\ \text{fully faithful}}}{\hookrightarrow} D(\mathrm{Bun}_G^{B\text{-gen}})$$

One can show $D(\mathrm{Bun}_G^{B\text{-gen}}) \hookrightarrow D(\mathrm{Bun}_G^{N\text{-gen}})$

Question: Why is $D(\mathrm{Bun}_G^{N\text{-gen}})$ easier given that $\mathrm{Bun}_G^{N\text{-gen}}$ is NOT Artin stack in general

BG is artin:

$$\cdots\, G \times G \times G \;\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\; G \times G \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; G \rightarrow 1$$

"colimit of Affine derived schemes w/ smooth morphisms"

Answer:
$$\begin{tikzcd}
\exists\, \mathrm{Whit}(G) \arrow[r, hook] & D(\mathrm{Bun}_G^{N\text{-gen}}) \arrow[d, phantom, "\circlearrowleft", shift right=8ex] \\
& D(\mathrm{Bun}_G) \arrow[u, hook] \arrow[ul, bend left, "!"]
\end{tikzcd}$$

factorization structure

Whittaker coefficient

1) Factorization algebras

$\mathrm{Gr}_{G,x}$ affine Grassmannian

$x \in C$

In terms of functor of points

$x : S \to C$

$\mathrm{Gr}_{G,x}(S)$

$$= \left\{ \begin{array}{l} G\text{-bundles on } C_S = C\times S \\ \text{w/ trivialization on } (C\setminus\{x\})\times S \\ \qquad\qquad\qquad\qquad\quad \shortparallel \\ \qquad\qquad\qquad\qquad C\times S \setminus \Gamma_x \end{array} \right\}$$

Prestack

$(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{Spc}$

$S \mapsto y(S)$

$\uparrow$ test scheme

Recall previously $\mathrm{Gr}_G = G(k((t)))/G(k[[t]])$

Define $\mathrm{Gr}'_{G,x}$ by:

$$\mathrm{Gr}'_{G,x} := \left\{ \begin{array}{l} G\text{-bundles on } \mathbb{D}_S \\ \text{w/ trivialization on } \mathbb{D}_S^{\times} \end{array} \right.$$

for $S = \mathrm{Spec}\, k$

$\mathbb{D} = \mathrm{Spec}\, k[[t]]$

$\mathbb{D}^{\times} = \mathrm{Spec}\, k((t))$

$\mathbb{D}_S := \mathrm{Spec}\, A[[t]]$

$\mathbb{D}_S^{\times} := \mathrm{Spec}\, A((t))$

Thm 1 (Beauville-Laszlo)

$$\mathrm{Gr}_{G,x} \to \mathrm{Gr}'_{G,x}$$

is an isomorphism

$$\Rightarrow \quad Gr_{G,x}(k) = Gr_{G,x}(\operatorname{Spec} k)$$
$$= G(k((t)))/G(k[[t]])$$

$$\begin{tikzcd} Gr_{G,x} \arrow[r] \arrow[d, "\Gamma"] & Gr_{G,C} \arrow[d] \\ x \arrow[r] & C \end{tikzcd}$$

Idea of Beilinson - Drinfeld:

can think about

BD Grassmannian $\rightarrow Gr_{G,C^n}$ $\downarrow$ C^n

$$C^n(S) = \{(x_1, \dots x_n) : S \to C\}$$

$$Gr_{G,C^n} = \left\{ \begin{matrix} (x_1, \dots x_n) : S \to C \\ G\text{-bundle on } C_S \\ \text{w/ triv. on } C_S / \Gamma_x \end{matrix} \right\}$$

$$\Gamma_x = \cup \Gamma_{x_i}$$

clear

look at $n=2$

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & Gr_{G,C^2} \arrow[d] \\ (x,y) \arrow[r] & C^2 \end{tikzcd}$$

① if $x=y$, it is $Gr_{G,x}$

② if $x \neq y$, it is $Gr_{G,x} \times Gr_{G,y}$

Heuristic: P_G G-bundle on C w/ triv on $C \setminus \{x,y\}$

$\leadsto$ P'_G on $C \setminus \{x\}$ $\Leftrightarrow$ P_G on D w triv on $D \setminus \{x\}$

P''_G on $C \setminus \{y\}$ $\qquad \Updownarrow$

P_G on $C \setminus \{\cancel{x}y\}$ w/ triv on $C \setminus \{x,y\}$

For general $\overset{n}{I} \xrightarrow{f} J$ surjective map of finite sets

$$\Delta_f : C^J \to C^I$$
$$(c_j)_{j \in J} \to (c_i)_{i \in I}$$

(Rem)

$$\begin{tikzcd} Gr_{G,C^J} \arrow[r] \arrow[d] & Gr_{G,C^I} \arrow[d] \\ C^J \arrow[r] & C^I \end{tikzcd}$$

② [Factorization] $I = I_1 \sqcup I_2$

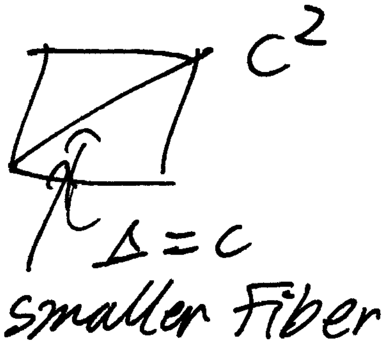
$$\begin{tikzcd} (Gr_{G,C^{I_1}} \times Gr_{G,C^{I_2}}) \times_{C^I} (C^{I_1} \times C^{I_2})_{disj} \arrow[r] \arrow[d] & Gr_{G,C^I} \arrow[d] \\ (C^{I_1} \times C^{I_2})_{disj} \arrow[r] & C^I \end{tikzcd}$$

where $C^{I_1} \times C^{I_2}{}_{disj} = \{ c_i \neq c_j \ \ \forall i \in I_1, \ \forall j \in I_2 \}$

Rmk] $Gr_{G,C^I} \to C^I$ formally smooth

$\uparrow$

ind-scheme of
ind-finite type
ind-proper for G reductive



Defn] D-space over X (D_X-space)
is an object of $PreStk/X_{dR}$

$$\begin{tikzcd} Gr_{G,C^I} \arrow[r] \arrow[d] & Gr_{G,C^I_{dR}} \arrow[d] \\ C^I \arrow[r] & C^I_{dR} \end{tikzcd}$$

$\leftarrow$ D-space over C^I
because

$X_I : S \to C$
$\Leftrightarrow X_I^{red.} : S^{red} \to C$

Defn| A factorization space over C is an assignment

$$I \to \mathcal{Y}_I \in PreStk/C^I_{dR}$$

satisfying the Ran axiom and the Factorization axiom

Ex| $Gr_{G, C^I_{dR}}$ is a fact. space

Factorization algebra
~ linearization of factorization space

Defn| A factorization algebra is an assignment

$$I \to A_{C^I} \in D(C^I) = QC(C^I_{dR})$$

s.t. ① $\forall I \overset{f}{\twoheadrightarrow} J$ $\quad \Delta f : C^J \to C^I$

$$\Delta_f^! A_{C^I} \simeq A_{C^J}$$

② (factorization)

$$A_{C^I}|_{(C^{I_1}\times C^{I_2})_{disj}} = (A_{C^{I_1}} \boxtimes A_{C^{I_2}})|_{(C^{I_1}\times C^{I_2})_{disj}}$$

for $I = I_1 \sqcup I_2$

$\mathcal{F} \in D(X) \quad \mathcal{G} \in D(Y)$

$$\mathcal{F} \boxtimes \mathcal{G} := \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{G}$$

where $\pi_1 : X \times Y \to X$
$\pi_2 : X \times Y \to Y$

Ex $I \longmapsto \omega_{C^I}$ is a factorization algebra

[Recall $\mathcal{X} \xrightarrow{p_\mathcal{X}} pt$, $\omega_\mathcal{X} := p_\mathcal{X}^! k$]

① $$\begin{tikzcd} C^J \arrow[rr, "\Delta_f"] \arrow[dr] & \circlearrowright & C^I \arrow[dl] \\ & pt & \end{tikzcd}$$

② $\omega_{C^{I_1}} \otimes \omega_{C^{I_2}} = \omega_{C^I}$

More generally, given a factorization space $\{\mathcal{Y}_I\}$ over C, one can construct a fact. algebra $\mathcal{A}_{C^I} := \pi_{I,dR*}\, \omega_{\mathcal{Y}_I}$ where $\pi_I : \mathcal{Y}_I \to C^I$ provided $\mathcal{Y}_I$ is nice enough

↖ Borel-Moore homology

e.g. $\mathcal{Y}_I$ is ind-scheme of ind-finite type

Summary

fact $A(x_1 \cdots x_n) \simeq A_{x_1} \otimes \cdots \otimes A_{x_n}$

if $x_i \neq x_j$ $\forall i,j$

Ran $\Rightarrow$ A depends only on $\{x_1 \cdots x_n\} \subset C$ subset

all information is in $A_x \hookleftarrow I = \{pt\}$
together with collision data

Big picture (Interlude)

$$\begin{tikzcd}
D(\mathrm{Bun}_G^{N\text{-gen}}) \\
\mathrm{Fun}(N(K)\backslash G(\mathbb{A})/G(\mathbb{O})) \arrow[u, "\wr"]
\end{tikzcd}$$

$K = K(C)$
↑
function field

$\{\mathrm{Gr}_{G,C^I}\}_I \rightsquigarrow G(\mathbb{A})/G(\mathbb{O})$

---

2) Group actions on Categories

i) Sheaves of Categories (?)

ShvCat/y for a prestack y
↑
Goal: define this

$S = \mathrm{Spec}\, A$ affine derived scheme
$\mathrm{ShvCat}/S = QC(S)\text{-mod}(\mathrm{DG\,Cat})$

$QC(S) = (A\text{-mod}, \otimes)$

is a comm alg obj. in DGCat

classical: $A \in Alg = Alg(Vect) \leftarrow \exists \mu : A \otimes A \to A$

$M \in A\text{-mod} = A\text{-mod}(Vect) \leftarrow A \otimes M \to M$

now: $A\text{-mod} \in Alg(DGCat) \leftarrow$ $A\text{mod} \otimes A\text{-mod} \to A\text{-mod}$

$\mathcal{F} \in (A\text{-mod})\text{-mod}(DGCat)$ $\quad A\text{-mod} \otimes \mathcal{F} \to \mathcal{F}$

$\mathcal{F} \in (A\text{-mod})\text{-mod}(DGCat)$

$$\Leftrightarrow A \to HC(\mathcal{F}) \overset{\text{"}}{} End(id : \mathcal{F} \to \mathcal{F})$$

Def $ShvCat_{/Y} = \lim_{S \to Y} ShvCat_{/S}$

$S \to T \rightsquigarrow f^* \; ShvCat_{/T} \to ShvCat_{/S}$

$\mathcal{C} \longmapsto QC(S) \otimes_{QC(T)} \mathcal{C}$

$$\begin{tikzcd} & QC(Y)\text{-mod} \arrow[d] \\ \Gamma : ShvCat_{/Y} \arrow[ur, bend left] \arrow[r] & DGCat \end{tikzcd}$$

$\mathcal{C} \longmapsto \Gamma(Y, \mathcal{C}) = \lim_{S \xrightarrow{f} Y} \Gamma(S, f^* \mathcal{C})$

$$\begin{tikzcd} \Gamma : QC(Y) \arrow[r] & Vect \\ \mathcal{F} \arrow[r, mapsto] \arrow[d, bend right] & \Gamma(Y, \mathcal{F}) \\ \mathcal{O}(Y)\text{-mod} \arrow[ur, bend right] & \end{tikzcd}$$

[illegible]

$\Gamma(Y, \mathcal{F}) = \lim_{S \xrightarrow{f} Y}$ [illegible] $(S, f^* \mathcal{F})$

$$\begin{tikzcd}
\mathrm{ShvCat}_{/\mathcal{Y}} \arrow[r, "\sim"] & \mathrm{DGCat} \\
\cup & \\
\mathrm{QC}_{/\mathcal{Y}} \arrow[r, Leftrightarrow] \arrow[ddr, bend right] & \mathrm{QC}(S) \\
 & \in \mathrm{QC}(S)\text{-mod}(\mathrm{DGCat}) \\
 & \{\mathrm{QC}(S)\} \otimes \mathrm{QC}(\mathcal{Y})
\end{tikzcd}$$

$$\mathcal{O}_{\mathcal{Y}} \Leftrightarrow \{\mathcal{O}_S\}_{S \to \mathcal{Y}}$$

$$\mapsto \mathcal{O}_{\mathcal{Y}}$$

Recall $\mathcal{Y}$ is called 0-affine if $\Gamma : \mathrm{QC}(\mathcal{Y}) \to \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})\text{-mod}$ is an equivalence

Defn (Gaitsgory) $\mathcal{Y}$ is called 1-affine if $\Gamma : \mathrm{QC}/\mathcal{Y} \to \mathrm{QC}(\mathcal{Y})\text{-mod}(\mathrm{DG}\text{-cat})$ is an equivalence

Ex/
- quasi-separated quasi-compact schemes
- Artin stacks of almost finite type
- For S of finite type, S_{dR}

---

non-example:
- $\mathbb{A}^\infty = \varinjlim \mathbb{A}^n$

Defn] $G\text{-}cat := ShvCat_{/\mathbb{B}G_{dR}}$

why? $G\text{-}rep = Rep\,G = QC(\mathbb{B}G)$

$\downarrow$

$(\rho, V) \longmapsto V$ underlying vector space

$G \circlearrowright_{\rho} \bullet \qquad G \circlearrowright V$

$\searrow$ V^G invariants

$\mathbb{B}G \xrightarrow{p} \{pt\}$

i: $\Gamma : QC(\mathbb{B}G) \to Vect$

$(\rho, V) \longmapsto \Gamma(\mathbb{B}G, V) = V^G$

ii: $\pi : pt \to pt/G$

$QC(\mathbb{B}G) \xrightarrow{\pi^*} Vect$

$(\rho, V) \to V$

$$\begin{array}{ccc} ShvCat_{/\mathbb{B}G_{dR}} & \to & DG\,cat \\ \mathcal{C} & \longmapsto & \Gamma(pt, \pi^* \mathcal{C}) \\ & & \shortparallel \\ & & \mathcal{C}^G \end{array}$$

$$\begin{array}{ccc} \Gamma \quad ShvCat_{/\mathbb{B}G_{dR}} & \to & DG\,cat \\ \mathcal{C} & \longmapsto & \Gamma(\mathbb{B}G_{dR}, \mathcal{C}) \\ & & \shortparallel \\ & & \mathcal{C}^G \end{array}$$

$\mathbb{B}G_{dR}$ is not 1-affine

$\boxed{11}$ Whittaker Category and Fundamental Local Equivalence

There is an equivalence of factorization cat.s

FLE $\mathrm{Whit}(G)_c \simeq KL(\check{G})_{-\check{c}}$

where c is level of G

Review G-action on Categories

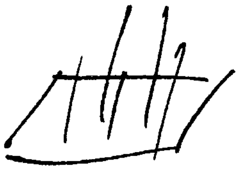
	classical (Vect)	categorical (DGCat)
algebra for S an affine derived scheme	$A \in \mathrm{Vect}$, ~~ex. $A = \mathcal{O}(S)$~~ $A \otimes A \to A$ $1 \in A$ ex. $A = \mathcal{O}(S)$	$\mathcal{C} \in \mathrm{DGCat}$ $\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ $1_{\mathcal{C}} \in \mathcal{C}$ ex. $\mathcal{C} = QC(S) = A\text{-mod}$
module	$M \in \mathrm{Vect}$ $A \otimes M \to M$ $\Leftrightarrow$ $A \to \mathrm{End}(M)$	$M \in \mathrm{DGCat}$ $\mathcal{C} \otimes M \to M$ $\Leftrightarrow$ $\mathcal{C} \to \mathrm{End}(M)$

Ex $M \in QC(S)\text{-mod} =: \mathrm{ShvCat}/S$

$$\begin{array}{l} QC(S) \\ \quad \| \\ (A\text{-mod})\text{-mod} \end{array}$$

e.g. $U\mathfrak{g}$-mod

$A\text{-mod} \to \mathrm{End}(M)$



Taking $\mathrm{End}(1)$

$$\begin{array}{ccc} A^{op} & \longrightarrow & \mathrm{End}(1_{\mathrm{End}(M)}) \\ \wr\wr & & \| \\ A & \longrightarrow & HC(M) \end{array}$$

	Classical	Categorical
prestack $\mathcal{Y}$	$QC(\mathcal{Y}) := \lim_{S\to\mathcal{Y}} QC(S)$	$ShvCat_{/\mathcal{Y}} := \lim_{S\to\mathcal{Y}} ShvCat_{/S}$
global section	$\Gamma: QC(\mathcal{Y}) \to \Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})\text{-mod}$ $\mathcal{F} \mapsto \lim_{S\xrightarrow{f}\mathcal{Y}} \Gamma(S, f^*\mathcal{F})$ $\mathcal{O}_{\mathcal{Y}} \mapsto \lim_{S\to\mathcal{Y}} \Gamma(S,\mathcal{O}_S)$ $\overset{\shortparallel}{\{\mathcal{O}_S\}}_{S\to\mathcal{Y}} =: \Gamma(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})$	$\Gamma: ShvCat_{\mathcal{Y}} \to QC(\mathcal{Y})\text{-mod}$ $\mathcal{C} \mapsto \lim_{S\to\mathcal{Y}} \Gamma(S, f^*\mathcal{C})$ $QC_{/\mathcal{Y}} \mapsto \lim \Gamma(S, f^*QC_{/\mathcal{Y}})$ $= \lim \Gamma(S, QC_{/S})$ $= QC(S)$
representation	$G\text{-rep} = \underset{\mathcal{F}\cup}{QC}(\mathbb{B}G)$	$G\text{-cat} = ShvCat_{/\mathbb{B}G_{dR}}$
underlying	$\pi: pt \to \mathbb{B}G$ $V = \pi^*\mathcal{F}$ is the underlying vector space	$\mathcal{V} \in G\text{-cat}$ $\pi_{dR}^*\mathcal{V} = \mathbb{V}$ is the underlying category
invariant	$V^G = \Gamma(pt, p_*\mathcal{F})$ $p: \mathbb{B}G \to pt$ $V^H = \Gamma(pt, p_* f^*\mathcal{F})$ $\mathbb{B}H \xrightarrow{f} \mathbb{B}G$ $\searrow pt \swarrow$	$\mathbb{V}^G = \Gamma(pt, p_{dR,*}\mathcal{V})$ $\mathbb{V}^H = \Gamma(pt, p_{dR,*} f_{dR}^*\mathcal{V})$

Ex $G \circlearrowright X$ $\qquad$ $\exists$ unique map $X \to pt$.

$$(X/G)_{dR} \xrightarrow{f} (\mathbb{B}G)_{dR}$$

$$\mathcal{V} = f_* QC_{(X/G)_{dR}} \in ShvCat_{/\mathbb{B}G_{dR}}$$

$$\begin{tikzcd}
X_{dR} \arrow[r] \arrow[d, "\pi'"] & pt \arrow[d, "\pi"] \\
X/\mathbb{G} \arrow[r] & \mathbb{B}\mathbb{G}
\end{tikzcd}$$

$$\begin{aligned}
V &= \Gamma(pt, \pi^* V) \\
&= \Gamma(pt, \pi^* f_* QC_{(X/\mathbb{G})_{dR}}) \\
&= \Gamma(pt, f'_* (\pi')^* QC_{(X/\mathbb{G})_{dR}}) \\
&= \Gamma(X_{dR}, (\pi')^* QC_{(X/\mathbb{G})_{dR}}) \\
&= \Gamma(X_{dR}, QC_{X_{dR}}) \\
&= D(X)
\end{aligned}$$

Ex. continued

$$\begin{aligned}
\mathbb{V}^{\mathbb{G}} &= \Gamma(pt, p_{dR,*} f_* QC_{(X/\mathbb{G})_{dR}}) \\
&\quad [(X/\mathbb{G})_{dR} \to B\mathbb{G}_{dR} \xrightarrow{p_{dR}} pt] \\
&= \Gamma((X/\mathbb{G})_{dR}, QC_{(X/\mathbb{G})_{dR}}) \\
&= D(X/\mathbb{G})
\end{aligned}$$

In other words, $D(X)^{\mathbb{G}} = D(X/\mathbb{G})$

$$\mathbb{G}\text{-cat} \xrightarrow{\Gamma} D(\mathbb{B}\mathbb{G})\text{-mod} \quad (DGCat)$$

$\parallel$

$ShvCat_{/\mathbb{B}\mathbb{G}_{dR}}$

would be an equivalence if $\mathbb{B}\mathbb{G}_{dR}$ is 1-affine

we don't want this, and indeed:

$\mathbb{B}\mathbb{G}_{dR}$ is not 1-affine (i.e. $ShvCat_Y \xrightarrow{\Gamma} QC(Y)$-mod $(DGCat)$ is not an equivalence)

$QC(\mathcal{Y}) \xrightarrow{\Gamma} \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$-mod is an equiv.

$\Leftrightarrow \mathcal{Y}$ is affine

$\mathrm{ShvCat}_{\mathcal{Y}} \to QC(\mathcal{Y})$-mod is an equiv.

$\Leftrightarrow \mathcal{Y}$ is 1-affine

$\mathbb{B}G_{dR}$ is not 1-affine but $\mathbb{B}G$ is

If you set up ~~re~~ cat. rep theory using $\mathbb{B}G$, not $\mathbb{B}G_{dR}$, then it is not interesting.

$$\begin{tikzcd} \mathbb{V}^{G} \arrow[r, shift left, "\mathrm{oblv}"] & \mathbb{V} \arrow[l, shift left, "\mathrm{AV}^{G}_{*}"] \end{tikzcd}$$

oblivion and averaging functors

<u>Ex</u> $\mathbb{V} = D(X)$ $\quad$ $\begin{tikzcd} D(X/G) \arrow[r, shift left, "\pi^*"] & D(X) \arrow[l, shift left, "\pi_*"] \end{tikzcd}$

$X \xrightarrow{\pi} X/G$

<u>Rmk</u> G-equivariancy is a datum, not a property. $\mathcal{F} \in D(X)^{G}$

$$\begin{tikzcd} G \times X \arrow[r, "act"] \arrow[d, "\pi"] & X \\ X & \end{tikzcd}$$

$\Leftrightarrow$ $\mathcal{F} \in D(X)$ w/ $\pi^*\mathcal{F} \simeq act^*\mathcal{F}$

In usual rep. theory, one has $V^{G} \subseteq V$

In particular, it makes sense to ask if $v \in V$ belongs to V^{G}.

Not any more in our setting.

On the other hand, if G is contractible ($e.g.\ \underset{\text{unipotent group}}{N \in B}$), then G-equivariancy is a property.

Khazhdan-Lusztig category

$$KL(G) \simeq (\hat{\mathfrak{g}}, G(\underset{\substack{\uparrow \\ \text{think of as} \\ G[[t]]}}{\mathcal{O}}))\text{-mod}$$

$\hat{\mathfrak{g}}$ is affine Kac-Moody algebra

$$= \text{central extension of } \underset{\substack{\cup \\ \mathfrak{g}[[t]]}}{\mathfrak{g}((t))}$$

so these are $(\mathfrak{g}, K)$ Harish-Chandra modules

but $\mathfrak{g} \to \hat{\mathfrak{g}}$ $\quad$ $\text{Lie}\, K \subseteq \mathfrak{g}$

$K \to G[[t]]$

What is $(\mathfrak{g}, K)$-mod in terms of DAG?

$$\begin{tikzcd} \text{DAG} \arrow[d, dashed] \\ \mathbb{B}K \leftrightarrow K\text{-mod} \end{tikzcd}$$

$\mathbb{B}G^{\wedge}_{\{1\}} \leftrightarrow \mathfrak{g}\text{-mod}$

Defn $(\mathfrak{g}, K)$-mod

$$= \text{IndCoh}(G^{\wedge}_{K})$$

Let $\mathcal{X} \to \mathcal{Y}$ map of prestacks

$$\mathcal{Y}^{\wedge}_{\mathcal{X}} = \mathcal{X}_{dR} \times_{\mathcal{Y}_{dR}} \mathcal{Y}$$

<u>Exer</u>^{1} $\mathcal{X} \hookrightarrow \mathcal{Y}$ is a clo
embedding, show
that this recover
classical notio

<u>Exer</u>^{2} Show $(\mathfrak{g}, K)$-mo
$\simeq \mathfrak{g}\text{-mod}^K$
using our definition

FLE: $Whit(G)_c \simeq KL(\check{G})_{-\check{c}}$

when $c=0$, $\check{c}=\infty$

$KL(\check{G})_\infty \simeq Rep(\check{G})$

obj are labeled by dominant wgts of $\check{G}$

= dominant wgts of G

<u>Now</u> What is $Whit(G)$?

$Whit(G) := D(Gr_G)^{N(K),\chi}$ $\quad K := \mathbb{C}((t))$

what is χ?

$$\begin{tikzcd} N(K) \arrow[r, "\chi"] \arrow[d, "ab"] & \mathbb{G}_a \\ (N/[N,N])(K) \arrow[r, "res"] & \prod \mathbb{G}_a \arrow[u, "\Sigma"] \end{tikzcd}$$

$G = GL_2$

$N = \left\{\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}\right\}$

$N(K) = \left\{\begin{pmatrix}1 & a \\ 0 & 1\end{pmatrix} \middle| a \in \mathbb{C}((t))\right\}$

$\exists$ exponential D-mod on $\mathbb{A}^1 = \mathbb{G}_a$
(aka Artin-Schrier sheaf)
in char p setting

$\chi = \chi^!(exp)$
$H \xrightarrow{\chi} \mathbb{G}_a$

$exp = D/D(\partial - 1)$ s.t. $add^!(exp) = exp \otimes exp$ $\Big\}$ multiplicative D-module

$add: \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$

$H \curvearrowright Y$ $\quad D(Y)^H \underset{Av_*}{\overset{oblv}{\rightleftarrows}} D(Y)$ $\quad$ for H, Y finite type

$\mathcal{F} \in D(Y)$

with $\quad act^*\mathcal{F} \cong \pi^*\mathcal{F}$

⋮

$$D(Y)^{H,\chi} \underset{Av_*}{\overset{oblv}{\rightleftarrows}} D(Y)$$

$\mathcal{F} \in D(Y)$

w/ $act^*\mathcal{F} \cong \pi^*\mathcal{F} \otimes \chi$

associativity ← need multiplicative nature of χ

---

what is $\quad D(Gr_G)^{N(K)}$?

① $\quad N(K) = \bigcup\limits_\alpha N_\alpha \qquad G = GL_2$

$N = \mathbb{G}_a$

$N(K) = \mathbb{C}((t))$

$N_\alpha = t^{-\alpha}\mathbb{C}[[t]]$

$\alpha \in \mathbb{Z}_{\geq 0}$

$$D(Gr_G) \supset D(Gr_G)^{N(K)} = \bigcap_\alpha D(Gr_G)^{N_\alpha}$$

② $\quad D(Gr_G)^{N_\alpha}$?

$Gr_G = \bigcup Y_\beta \quad \begin{cases} Y_\beta \text{ f.d.} \\ \text{inv. under } N_\alpha \end{cases}$

$$\Rightarrow D(Gr_G)^{N_\alpha} = \varprojlim_\beta D(Y_\beta)^{N_\alpha}$$

③ N_α is still ∞- dim!

$$N_\alpha = \lim_{\gamma} N_{\alpha,\gamma}$$

Ex $G = GL_2$

$N_{\alpha,\gamma}$ truncation of Taylor series.

Then $D(Y_\beta)^{N_\alpha} = D(Y_\beta)^{N_{\alpha,\gamma}}$

$\gamma >> 0$

$K = \mathbb{C}((t))$ $\quad \mathcal{O} = \mathbb{C}[[t]]$

Rmk/ consider

$$\begin{array}{ccc} D(N(K)/N(\mathcal{O}))^{N(K)} & \simeq & \mathrm{Vect} \\ & & \cup \\ \omega_{Gr_N} & \longleftrightarrow & 1 \end{array}$$

$G = GL_2$, $\Big($ $N = \mathbb{G}_a = \mathbb{A}^1$

$N(K)/N(\mathcal{O}) = \mathbb{A}^\infty = \bigcup \mathbb{A}^n$

as $\mathbb{A}^n$ is smooth

$\{\omega_{\mathbb{A}^n}\}_{n \in \mathbb{Z}_{\geq 0}}$ $\quad \omega_{\mathbb{A}^n} \simeq \mathcal{O}_{\mathbb{A}^n}[n]$

ω_{Gr_n} is a phantom object

↑ cohomologically trivial,

but non-zero!

$D(Gr_G)^{N(K)}$

Q: $N(K)$-orbits in Gr_G?

a coweight of G, $\lambda \in \Lambda_G$

$\mathbb{G}_m \to T \subset G$

one considers $\mathbb{G}_m(K) \to T(K) \subset G(K) \ni t \mapsto \lambda(t) \rightsquigarrow t^\lambda \in Gr(G)$

Claim $N(\mathcal{K})\cdot t^\lambda =: S^\lambda$

$\mathrm{Gr}^\lambda = G(\mathcal{O})\, t^\lambda$
$= G(\mathcal{O})\,\lambda(t)\, G(\mathcal{O})$

$\Rightarrow \mathrm{Gr}_G = \bigcup_{\lambda \in X_G} S^\lambda$ by Iwasawa decomposition

---

$G = GL_2$

$\lambda \in \Lambda^+_G$ (m,n) w/ $m \geq n$

$\lambda \in \check{\Lambda}_G$ (m,n)

Rmk (Geometric Satake)

$H(\mathrm{Gr}^\lambda_G) \simeq V^\lambda$

$\lambda \in \check{\Lambda}^+_G \leftrightarrow \lambda \in \Lambda^+_{\check{G}}$

$H^\bullet(S^\mu \cap \mathrm{Gr}^\lambda) \longleftrightarrow V^\lambda_\mu \subset V^\lambda$ (pretty amazing)

$D(S^\lambda)^{N(\mathcal{K})} = \mathrm{Vect}$ $\forall \lambda \in \check{\Lambda}_G$

Q: What about $D(S^\lambda)^{N(\mathcal{K}),\chi}$?

Claim $D(S^\lambda)^{N(\mathcal{K}),\chi} = \begin{cases} \mathrm{Vect} & \text{if } \lambda \in \check{\Lambda}^+_G \\ 0 & \text{otherwise} \end{cases}$

$i^\lambda : S^\lambda \hookrightarrow \mathrm{Gr}_G$

$$\begin{tikzcd} D(S^\lambda)^{N(\mathcal{K}),\chi} \arrow[r, shift left, "(i^\lambda)_!"] & D(\mathrm{Gr}_G)^{N(\mathcal{K}),\chi} \arrow[l, shift left, "(i^\lambda)^!"] \end{tikzcd} = \mathrm{Whit}(G)$$

$\mathrm{Whit}(G) \simeq KL(\check{G})_{op} = \mathrm{Rep}(\check{G})$

has objects labeled by

$\check{\Lambda}^+_G \simeq \Lambda^+_{\check{G}}$

If $H \circlearrowright Y$ transitive, $Y = H/H_1$

$H_1 = \mathrm{Stab}_y(H)$, $y \in Y$

$D(Y)^{H,\chi}$ contractible

$= D(pt)^{H_1, \chi|_{H_1}}$

$= \begin{cases} \chi|_{H_1} \text{ is trivial} \Rightarrow \mathrm{Vect} \\ \chi|_{H_1} \text{ is nontrivial} \Rightarrow 0 \end{cases}$ V w/ $\pi^* V \cong act^* V \otimes \chi|_{H_1}$

~~w/~~ $\pi = act : H_1 \times pt \to pt$

For us:

$H = N(K)$ $Y = Gr_G$ $y = t^\lambda$ $\lambda \in \check{\Lambda}_G$

$H_1 = \mathrm{Stab}_{t^\lambda}(N(K))$

$= \mathrm{Ad}_{t^\lambda} N(\mathcal{O})$

$\chi|_{H_1}$?

$G = GL_2$ $\lambda = (m,n)$ not necessarily dom $m \geq n$

$$\begin{pmatrix} t^m & 0 \\ 0 & t^n \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-m} & 0 \\ 0 & t^{-n} \end{pmatrix} = \begin{pmatrix} 1 & b^{m-n} \\ 0 & 1 \end{pmatrix} \to \chi$$

$\Rightarrow \chi = 0 \Leftrightarrow m \geq n$

$\Leftrightarrow \lambda$ is dominant

$$\Rightarrow D(S^\lambda)^{N(K),\chi} = \begin{cases} \mathrm{Vect} & \text{if } \lambda \in \check{\Lambda}^+_G \\ 0 & \text{otherwise} \end{cases}$$